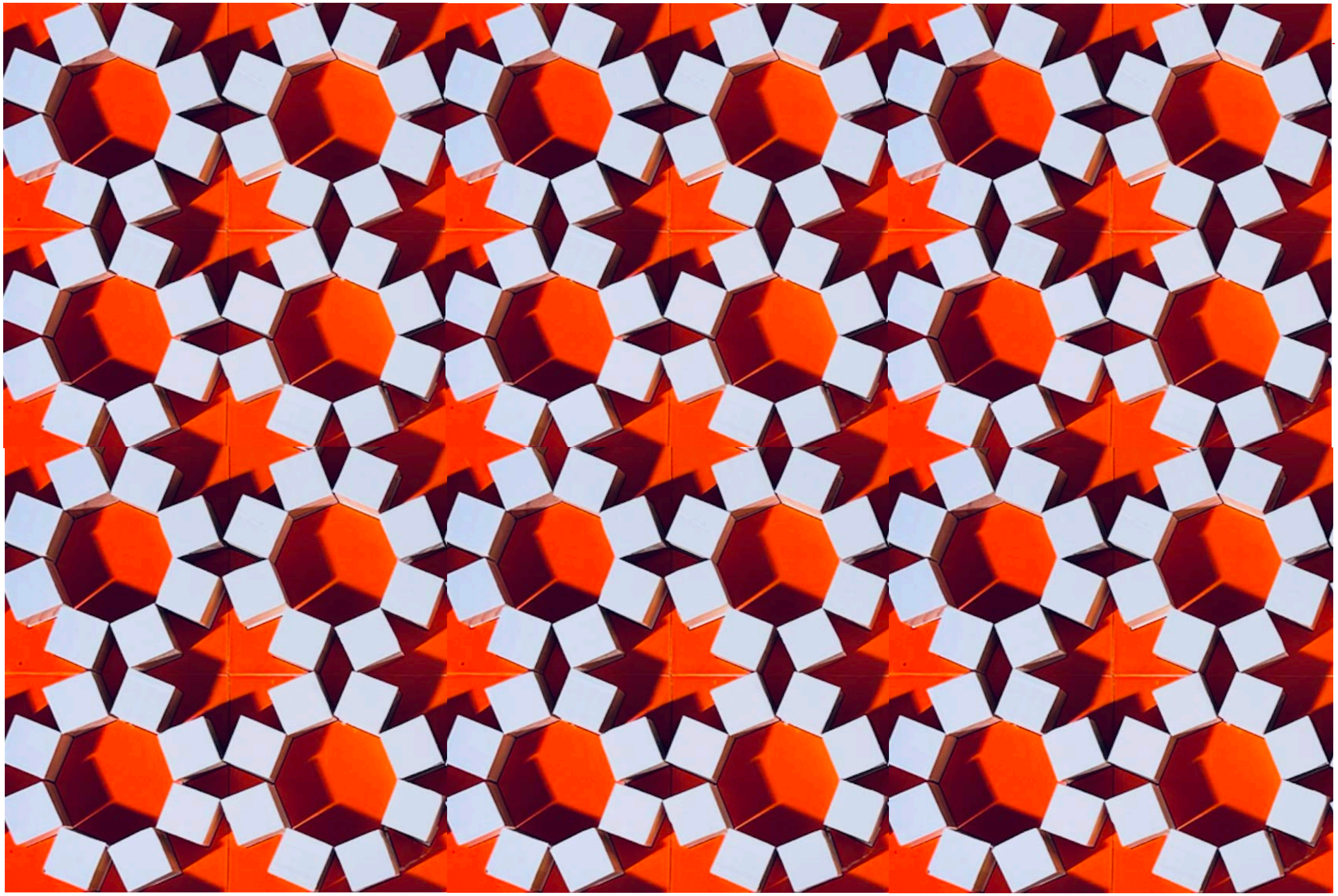
geometria2

“Square Dance“



• Symmetry operations on a square with Geometria2© • • Janos Baracs, Montréal 2025 •

M. C. Escher 1898 - 1972

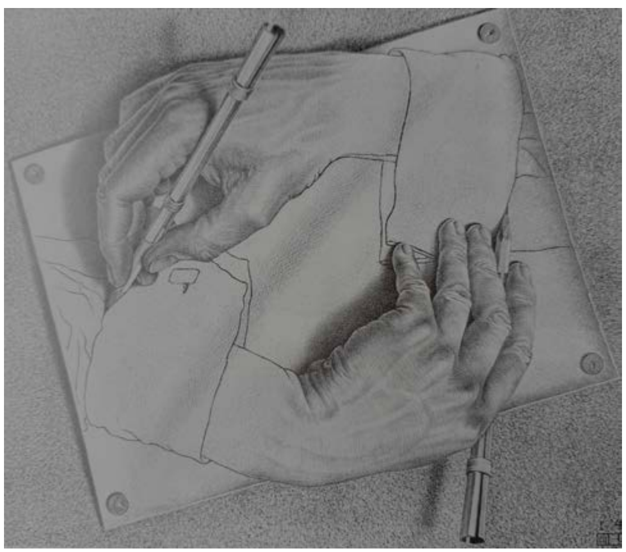
The world of mathematics and art owes much to Escher, the Dutch graphic artist. He first visited the Alhambra in 1922, the palace in Granada. This architectural masterpiece is now recognized as a true mathematical marvel. Its Moorish ornaments encompass all seventeen symmetry groups, which were not fully understood until the end of the 19th century. The wonderful prints of Escher on the regular division of the plane inspired these pages.

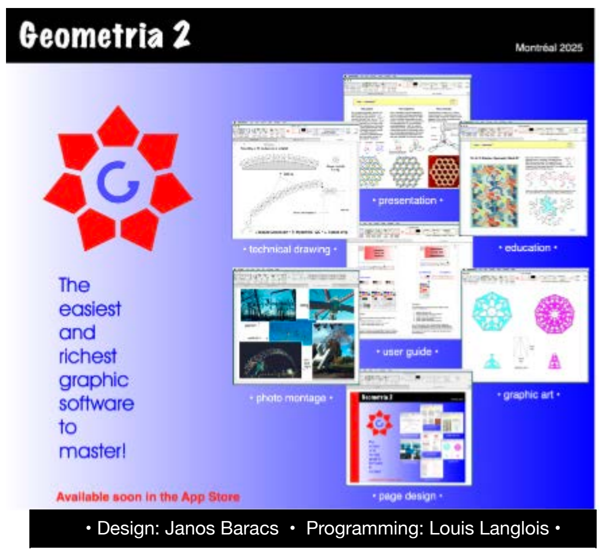
The software Geometria2©

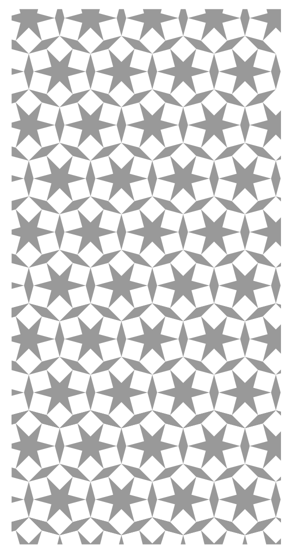
Today, it is exciting to be part of the digital revolution. We are near the completion of the drafting and educational software Geometria2, an efficient tool among other functions to create and analyze tiling and patterns. Unfortunately, symmetry is not part of the curriculum at any level in our educational system, and also absent in most graphic software. All pages of this document were prepared with the beta version of Geometria2 and we enjoyed using the program!

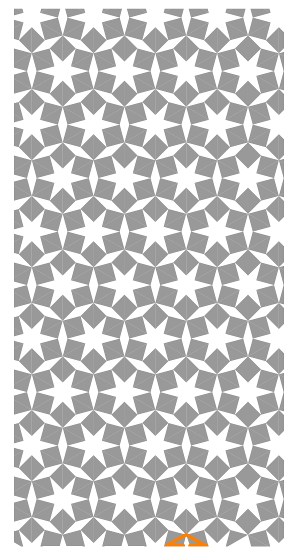
Square Dance, a brief study

The title, “Square Dance”, came from just looking at the patterns created by the rhythmic motion of a lowly square. A simple motif is placed inside a special triangle or a square, followed by repetitive reflections about the edges of the polygon, using a single tool of Geometria2. The first pages present an overview of the patterns. Page 6 and 7 explains the constructions, and page 8 shows the alternate colouring, followed by the sixteen patterns, we are aware of as of today.

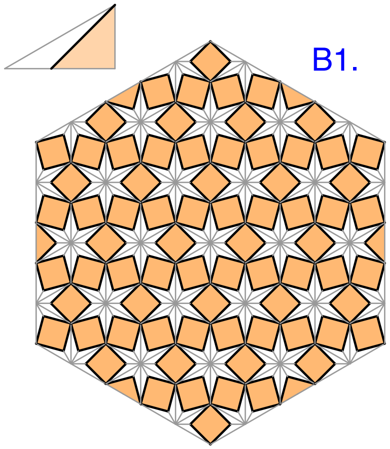








• A3 Twelve variations • Square Dance •



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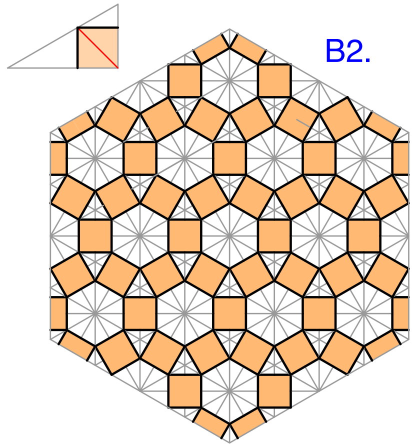

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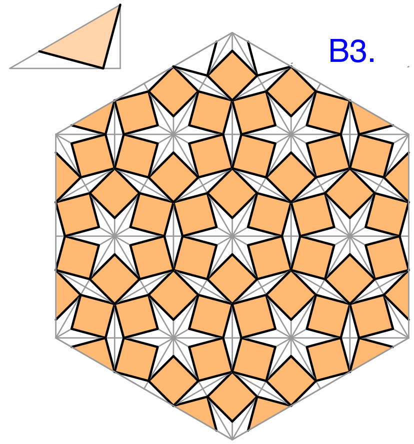

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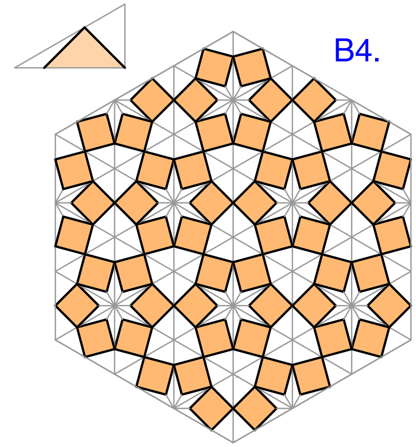

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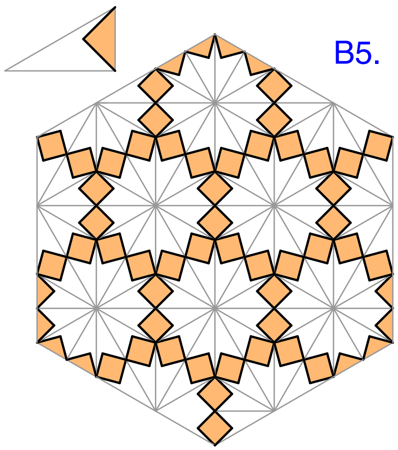

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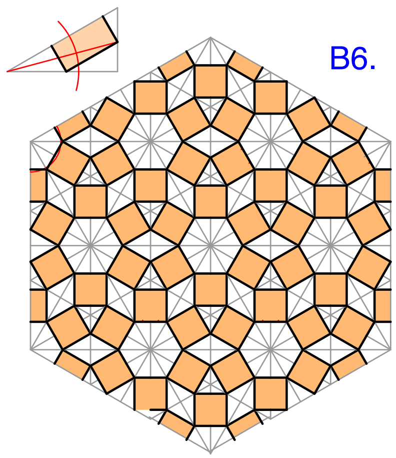

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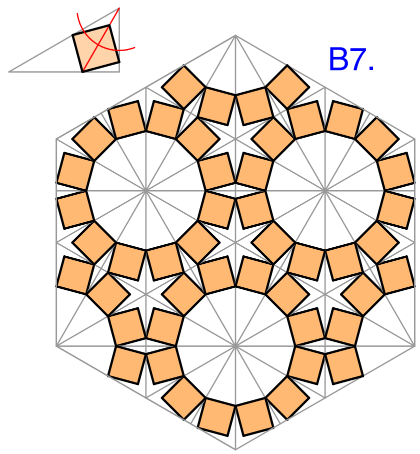

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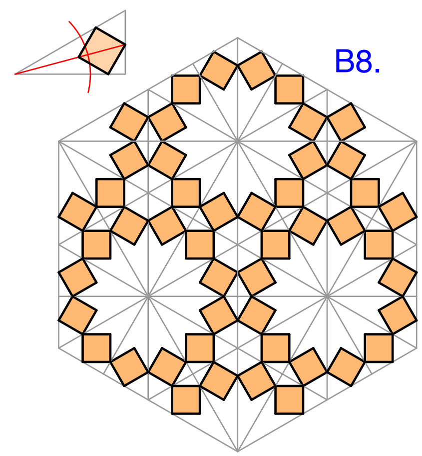

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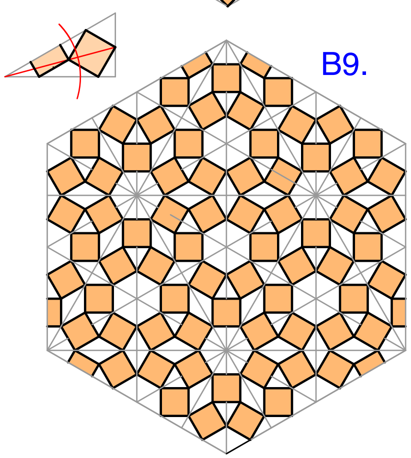

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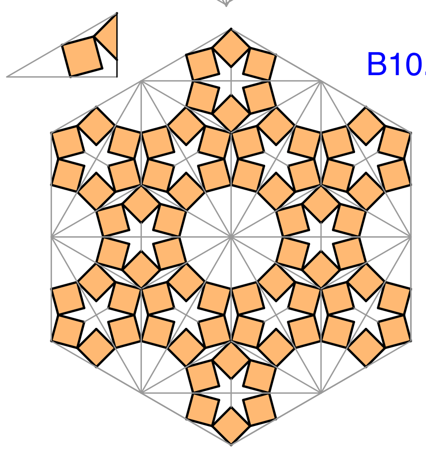

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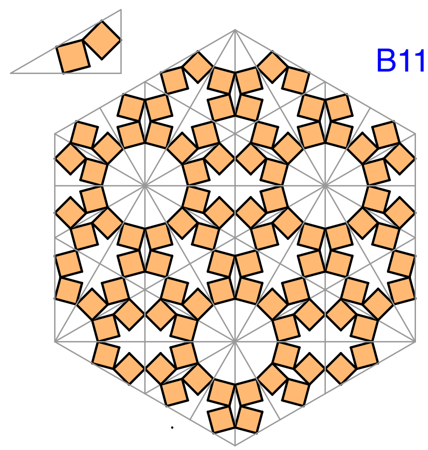

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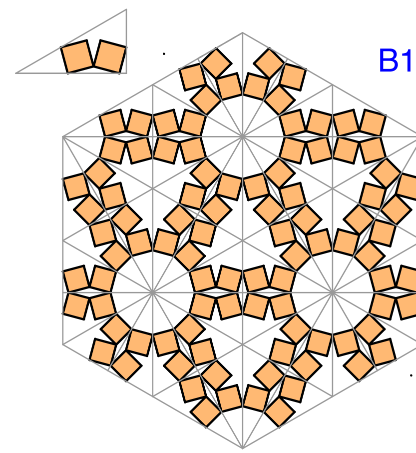
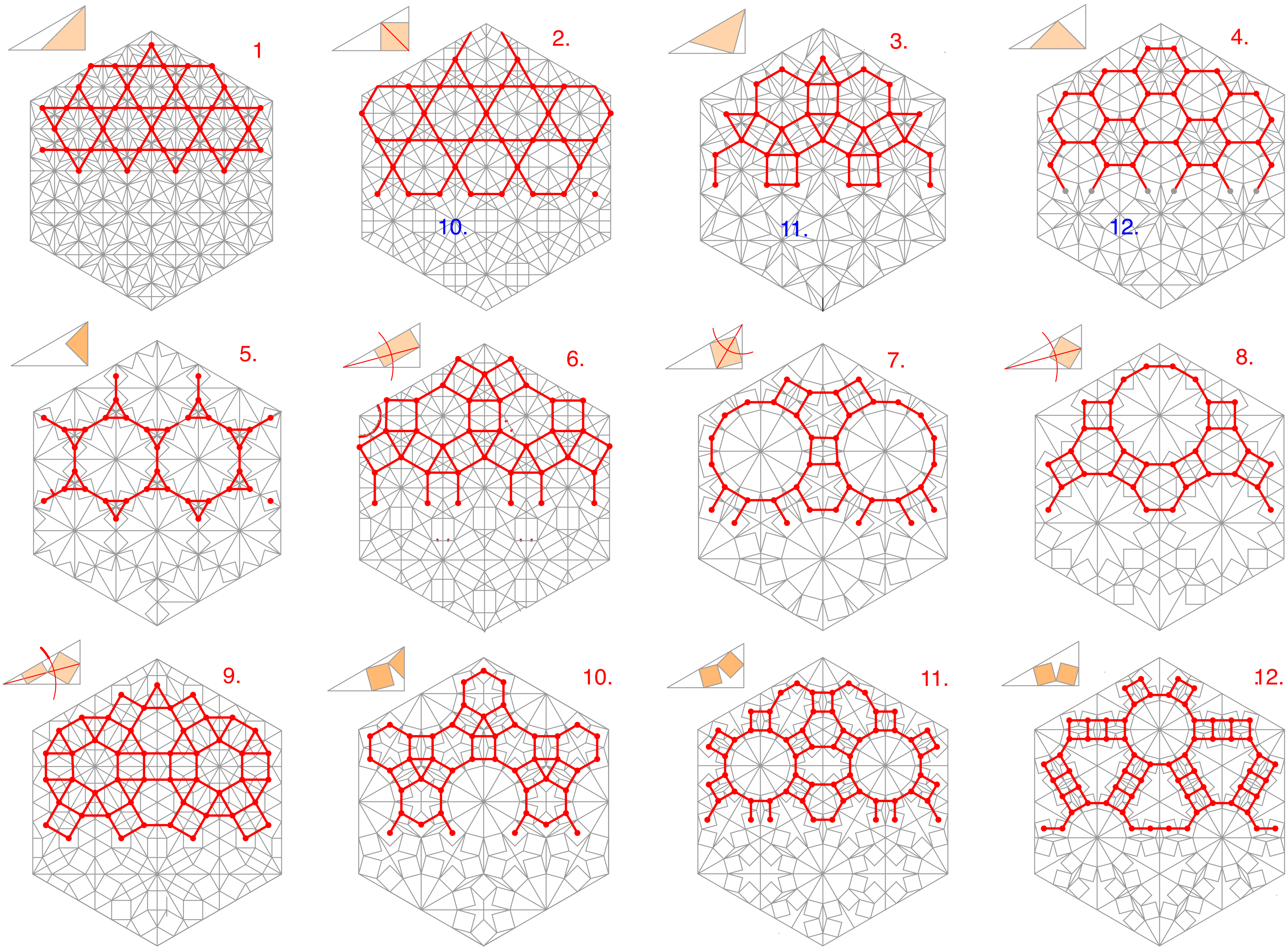


B12.

• A4 The dual graphs • • Square Dance •



• A5 Another view • • Square Dance •

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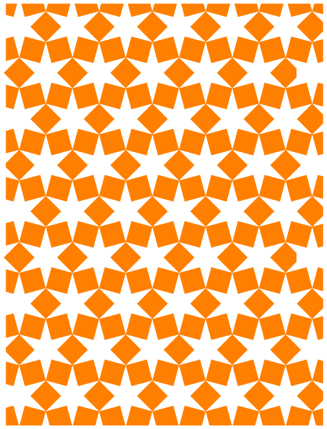

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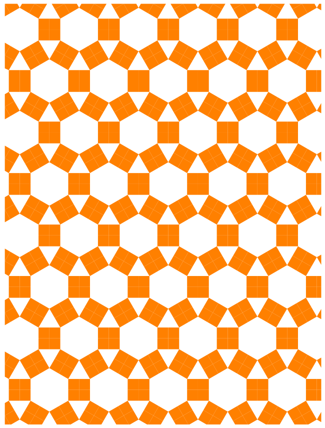

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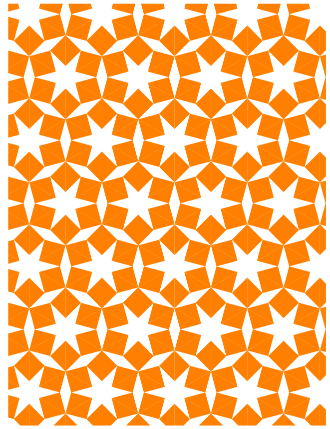

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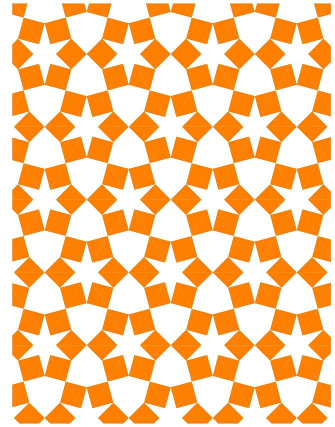

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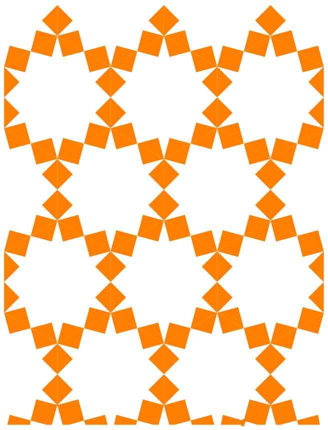

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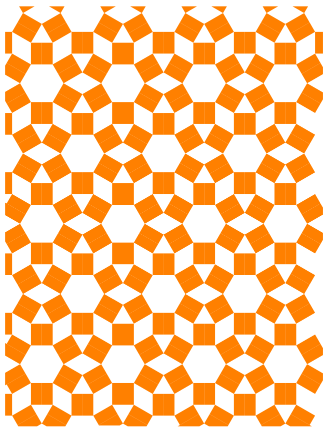

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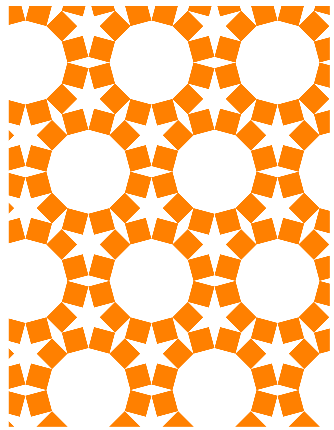

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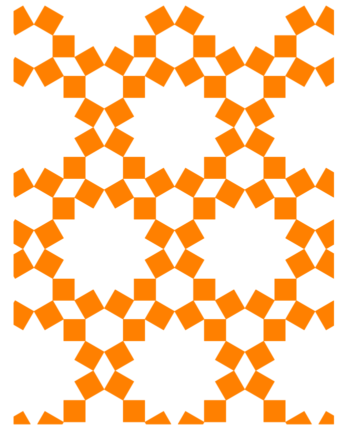

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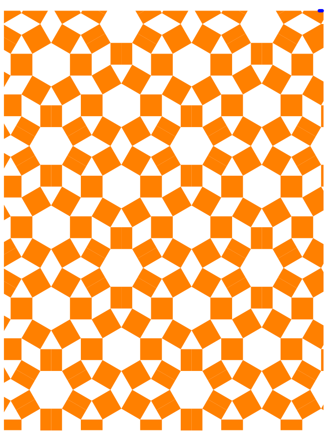

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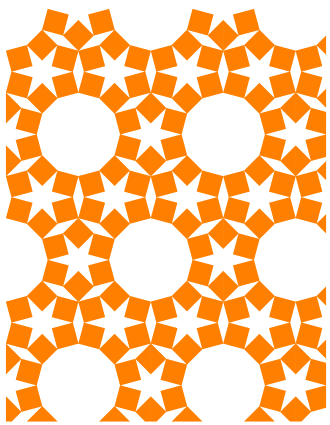

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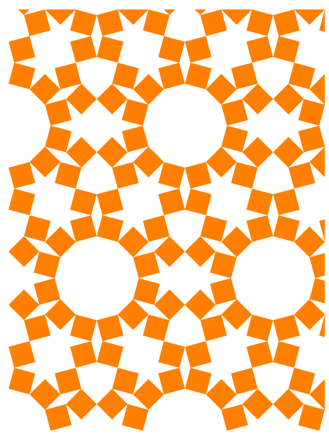

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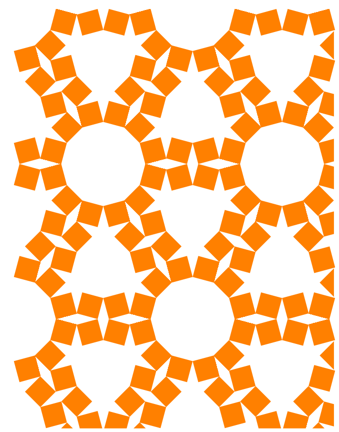
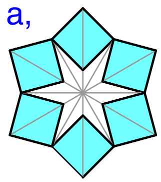
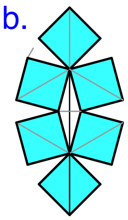


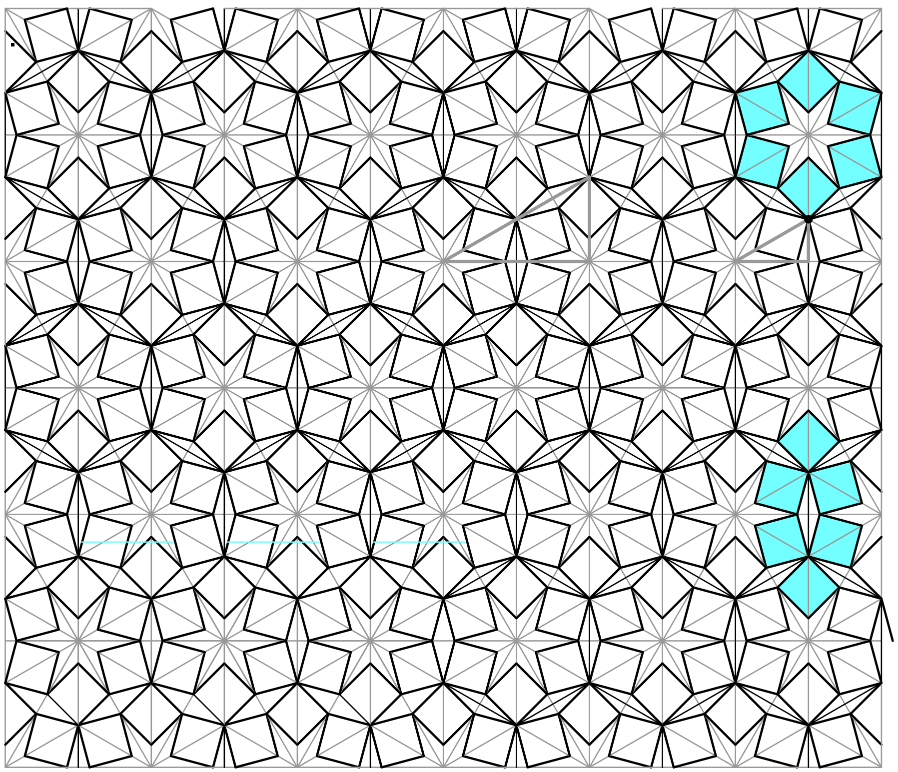
The #3 square pattern is selected to present two interesting properties. Like all others, it is generated by repetitive reflections about the sides of a half-equilateral triangle, displaying the symmetry group p6m. As mentioned earlier, we privileged this group over the other sixteen because of its rich and complex results. This is also the reason for including it among Geometria2's basic functions.
The fundamental region of a pattern is the smallest figure which can generate the pattern with translation only (parallel displacement). The pattern #6 has two distinct unit cells (a and b), each contains six squares.

a,

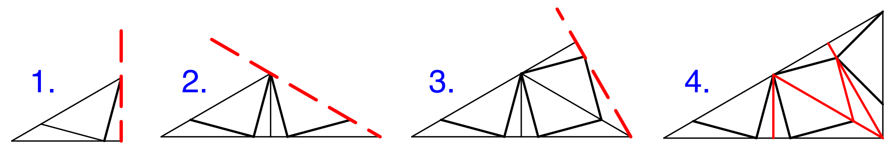


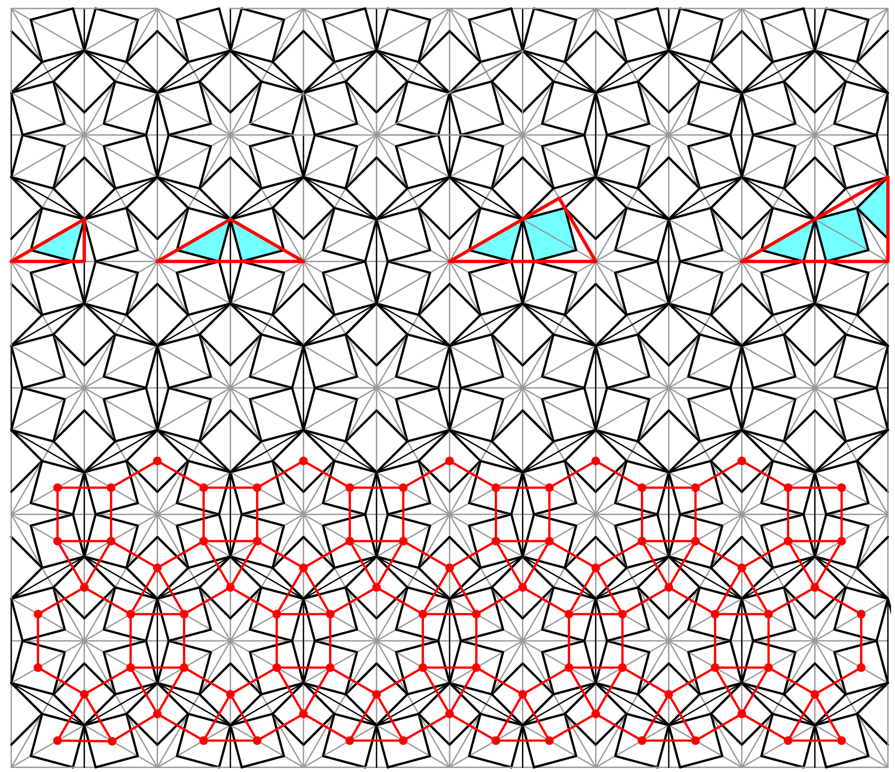
b.





The unit cell of a square pattern (a special triangle or a square) is subjected to repetitive reflections to generate the pattern. Another interesting property of many square patterns is that different unit cells may generate the same square pattern, containing different motifs inside.
In the example shown below, the unit cells are components of an equilateral (regular) triangle.
The #1 unit cell is one 30°-60° triangle, #2, #3, and #4 are two, three, and four 30°-60° triangles, respectively. The dual pattern (red lines, see Page 2) is at the bottom of the page.



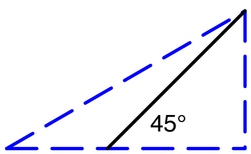


The drafting of the 16 patterns is divided into two groups:

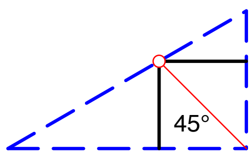
The drafting of the 9 patterns in the first group starts with drawing the triangles. For B1, B3, B4 and B5, an edge drawn at 45° is sufficient to draw the square(s). For B2, B6, B9, and B13, you need to draw an angle bisector. For B10, a 30° line starts the construction.

The drafting of the 7 patterns in the second group starts with drawing the square(s). There is a limited freedom to orient the square(s): no edge should be incident with the sides of the triangle. Then you draw three lines through three vertices with the correct angles with the same conditions.

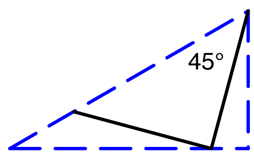
B1



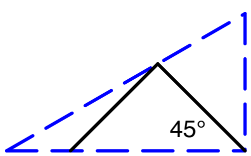
B2



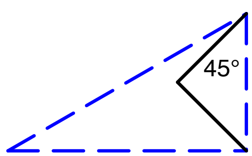
B3



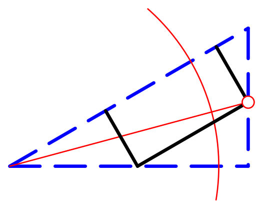
B4



B5



B6



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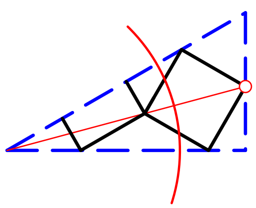

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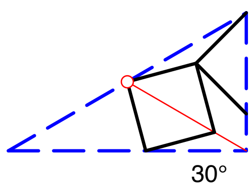

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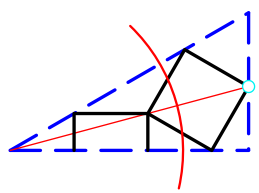

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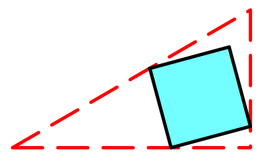

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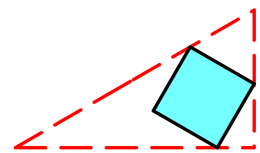

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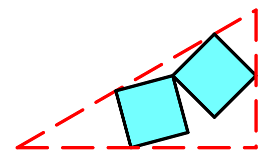

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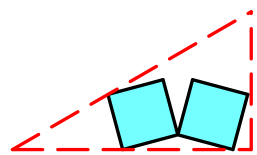

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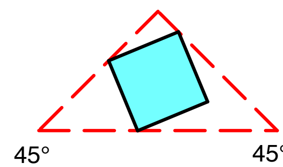

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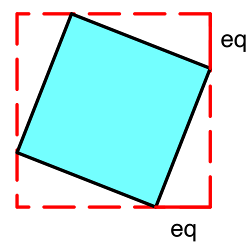
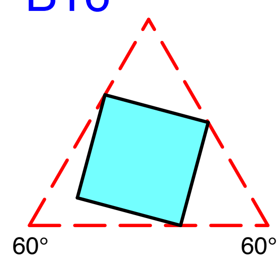
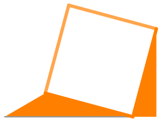


B16

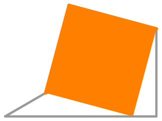


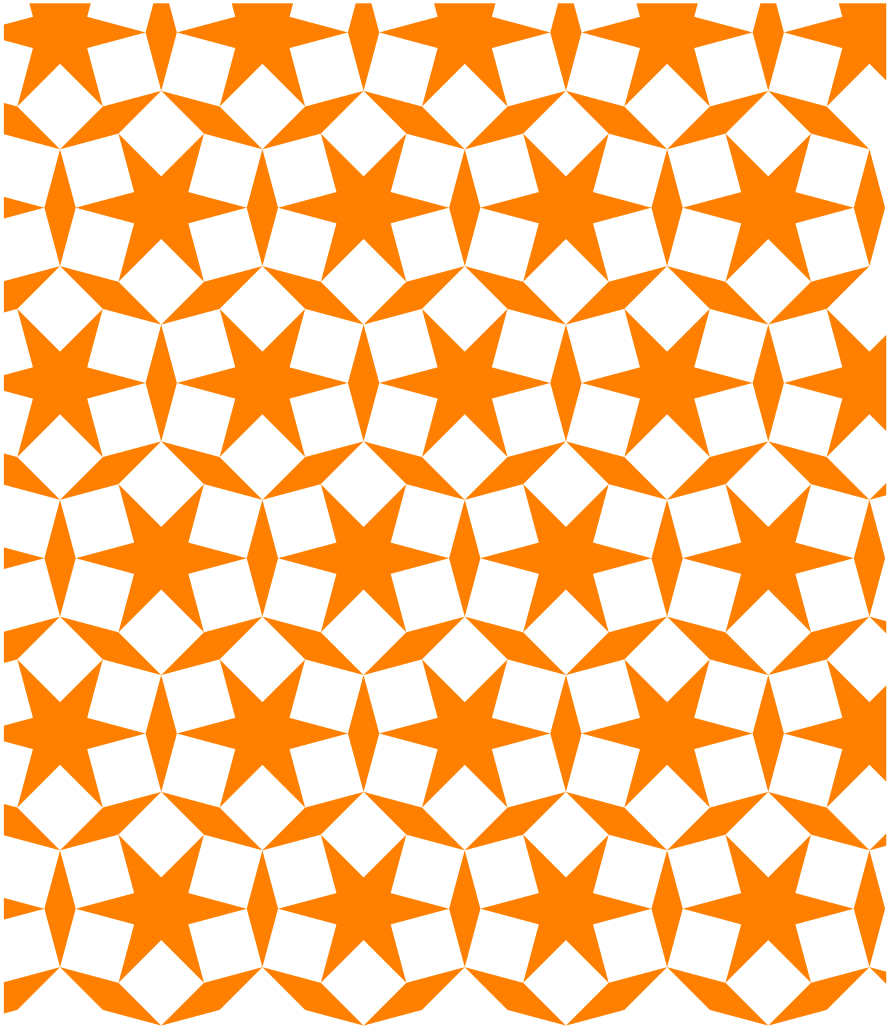
Rules for the square patterns:

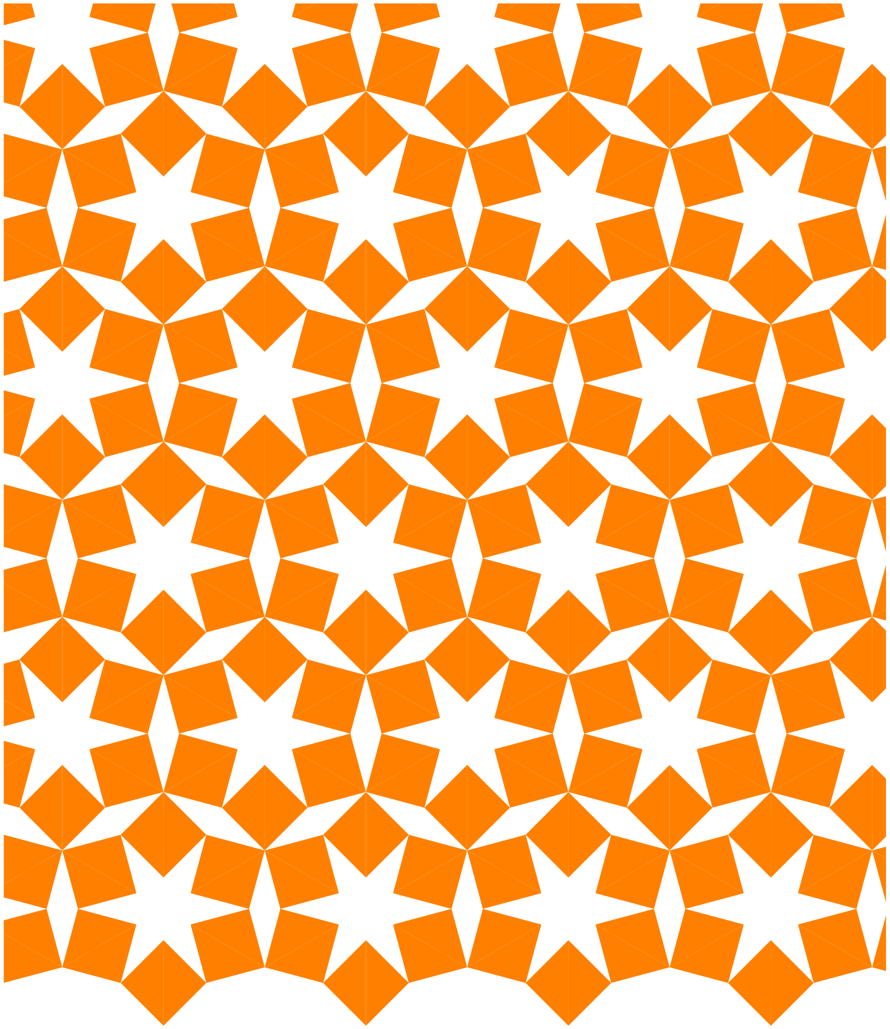
1. Squares are all of equal size
2. No intersections of squares
3. No two squares share an edge
4. Squares share at least two vertices
5. Maximum two squares per unit cell



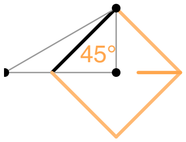
If the number of lines meeting at every vertex is an even number, two colours are sufficient to colour the pattern, such, that two adjacent polygons do not have the same colour. This also means that only two alternatives are available. The one on the left, the squares are white, and all other polygons are orange; on the right, the opposite is true. This applies to the sixteen patterns that follow, because all the vertices are 4- and 6-valent.

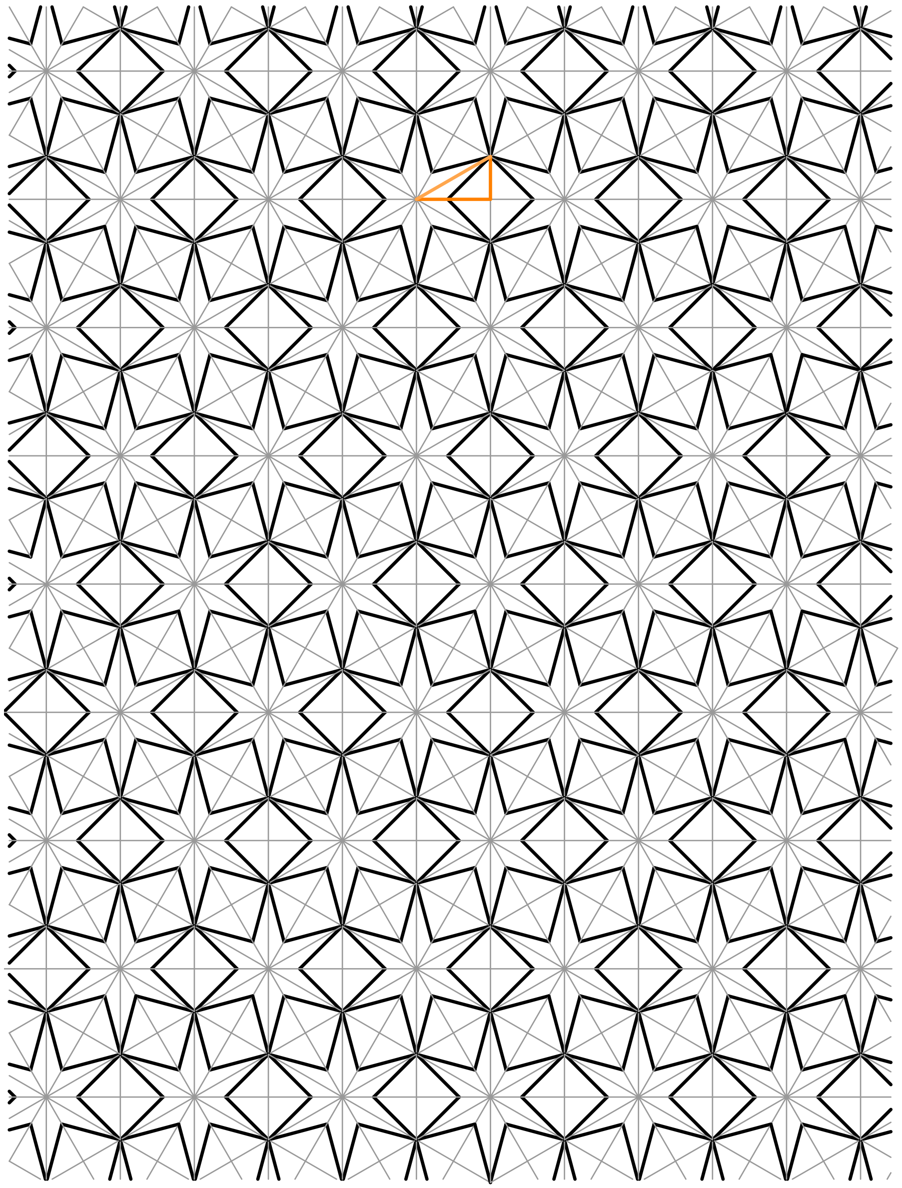


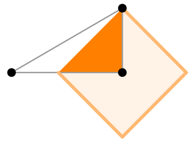


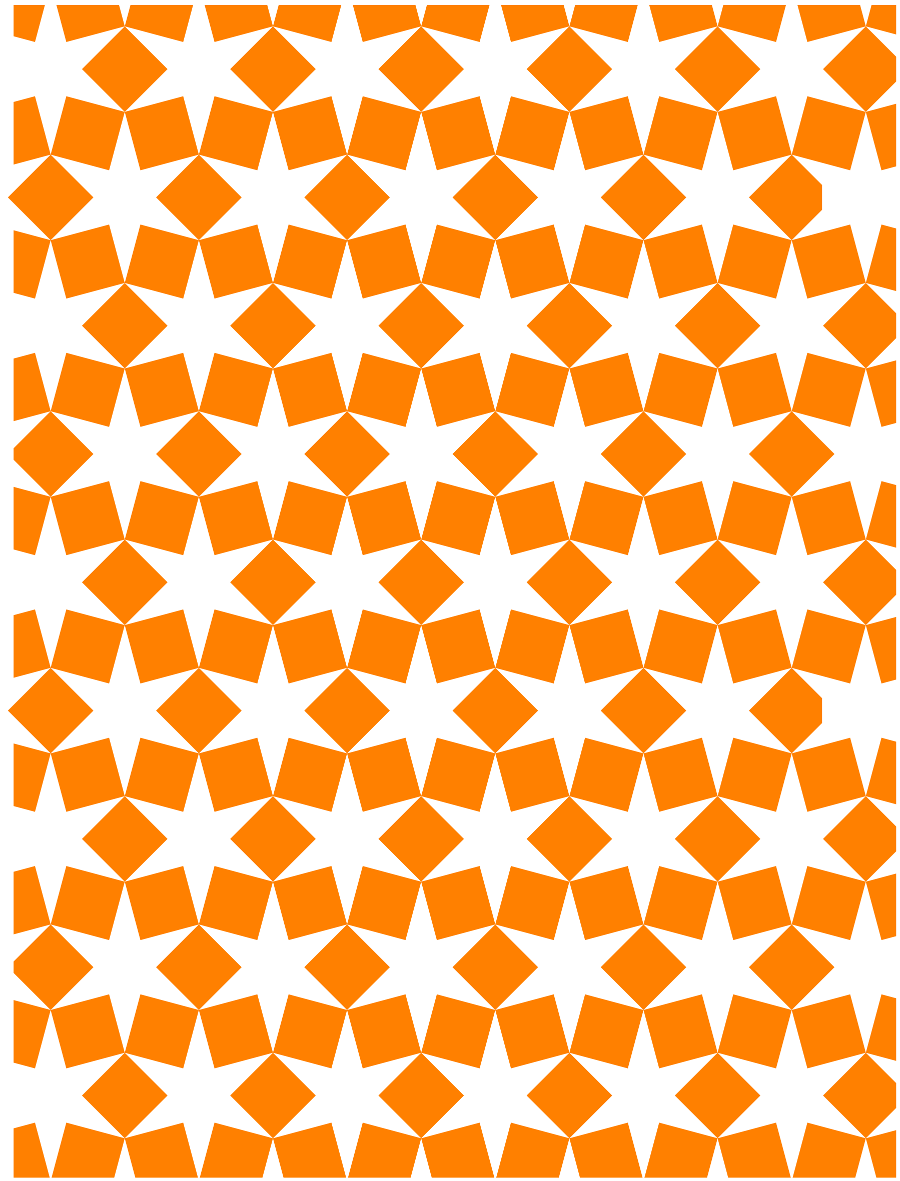


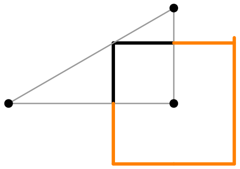
• B1 Quarter square in a triangle • • Square Dance •

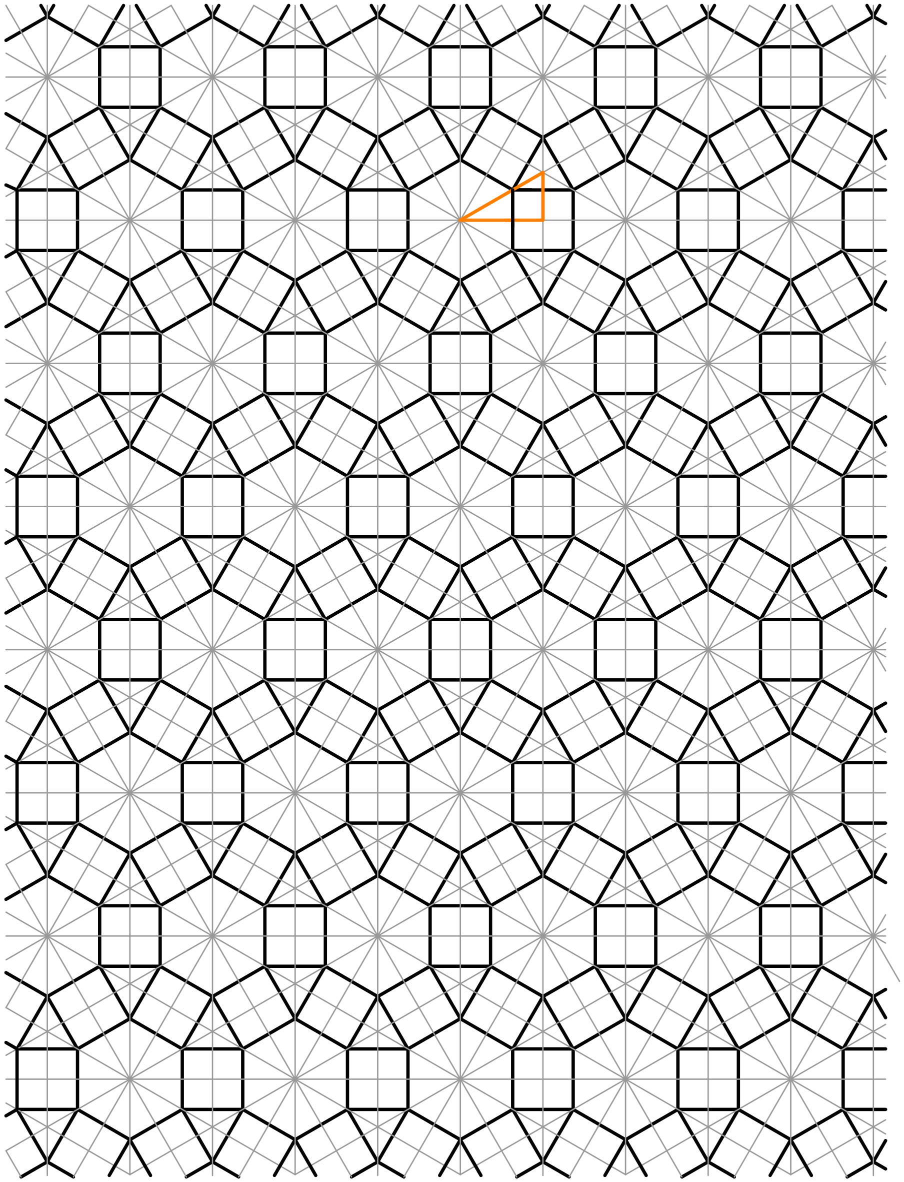


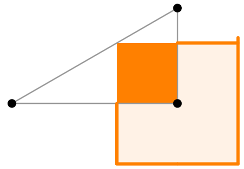


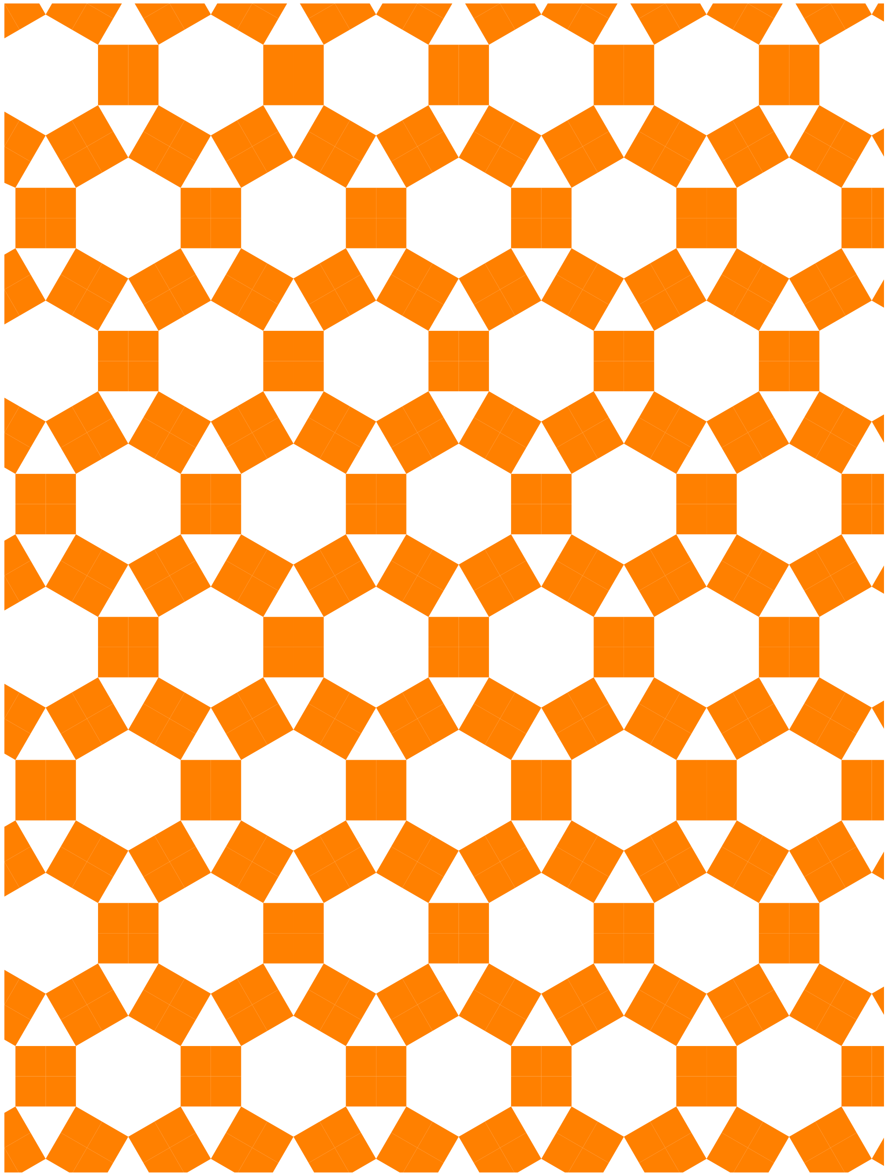




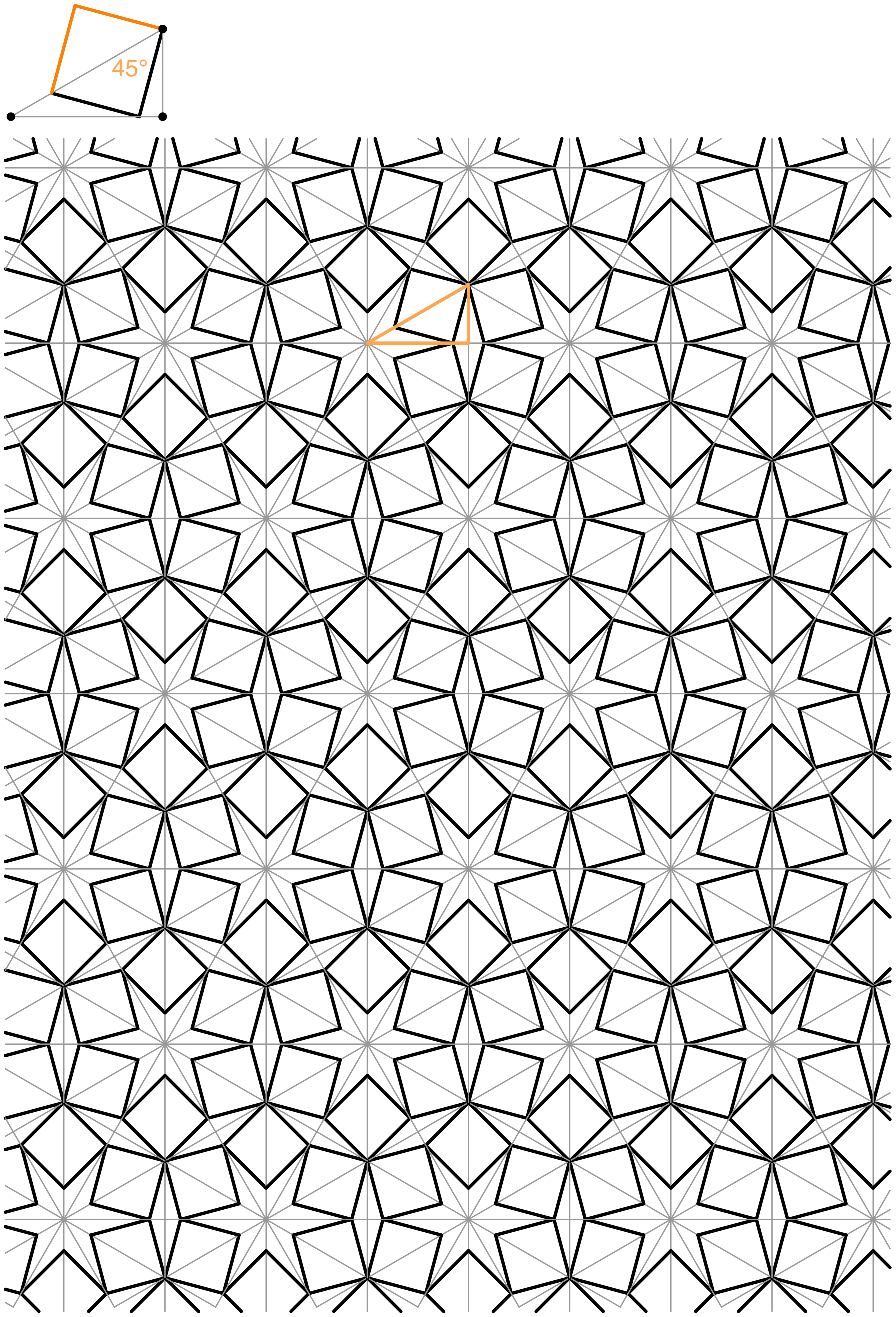


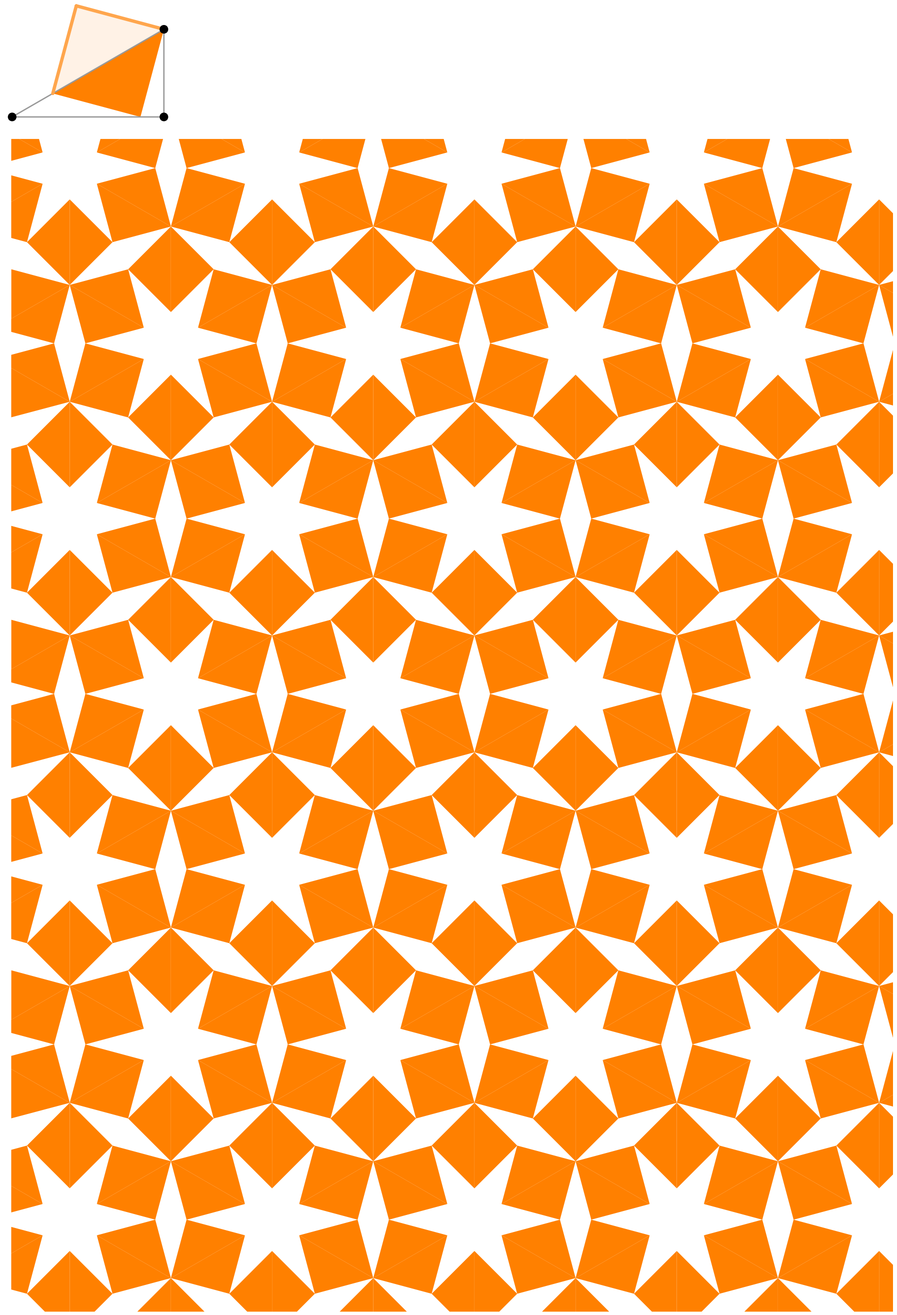




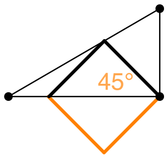


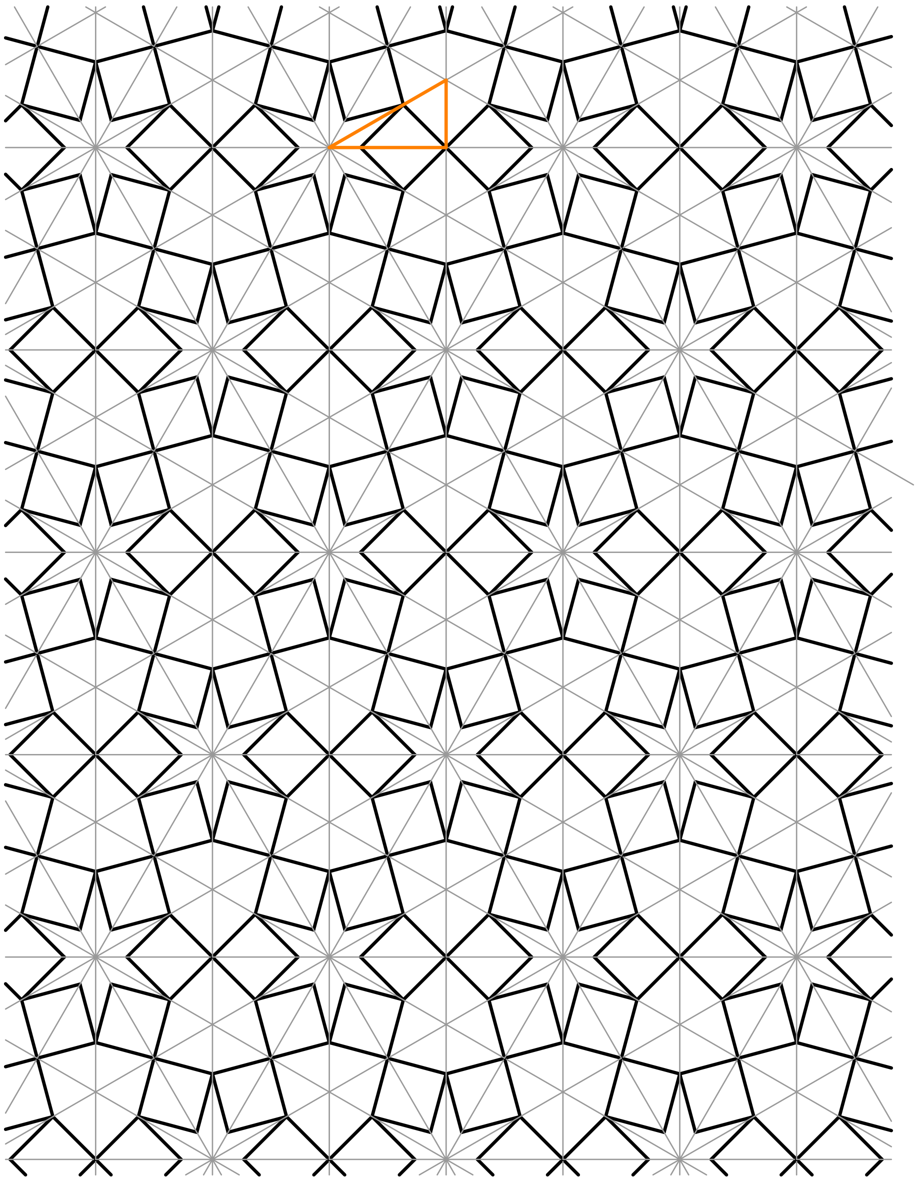
• B3 Half square in a triangle • • Square Dance •

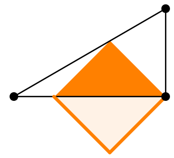


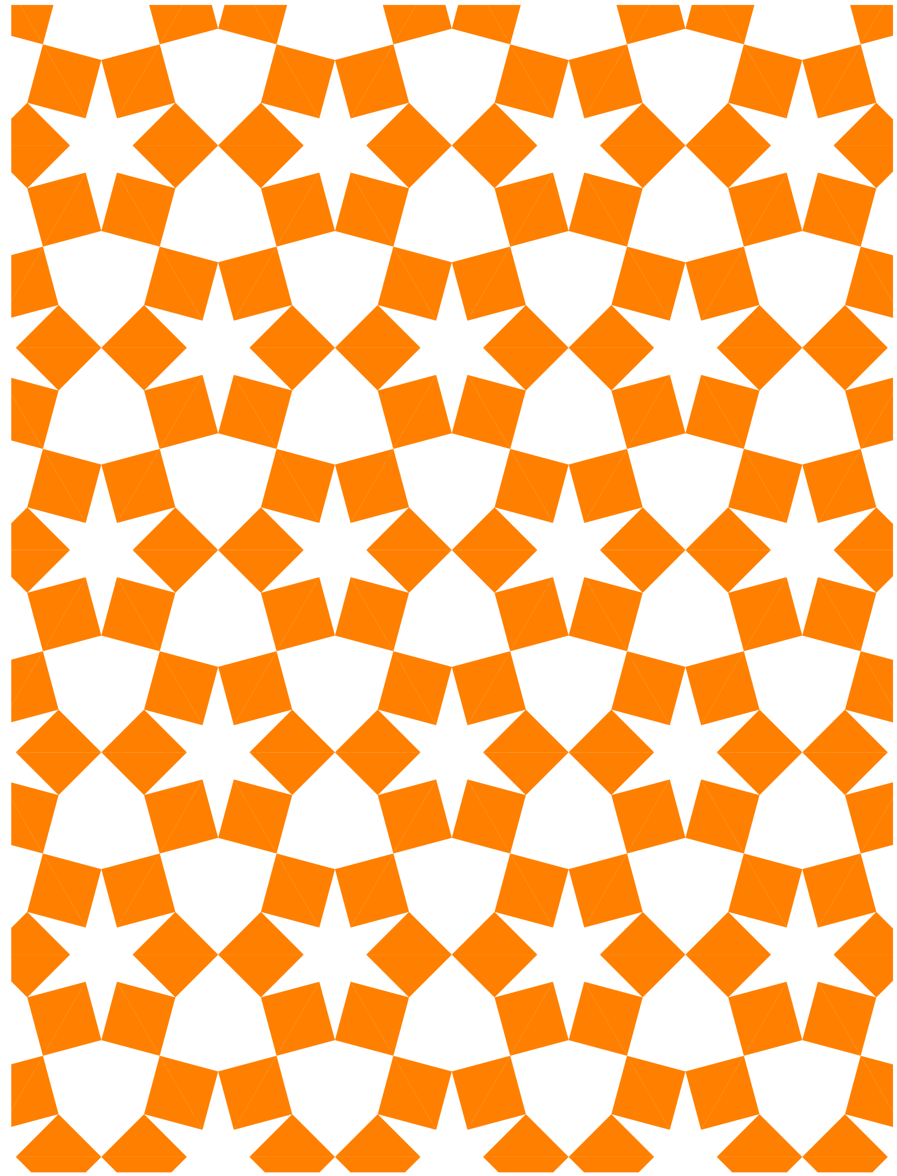


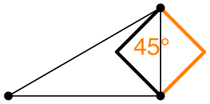
• B4 Half square in a triangle • • Square Dance •







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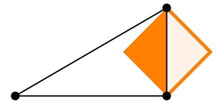

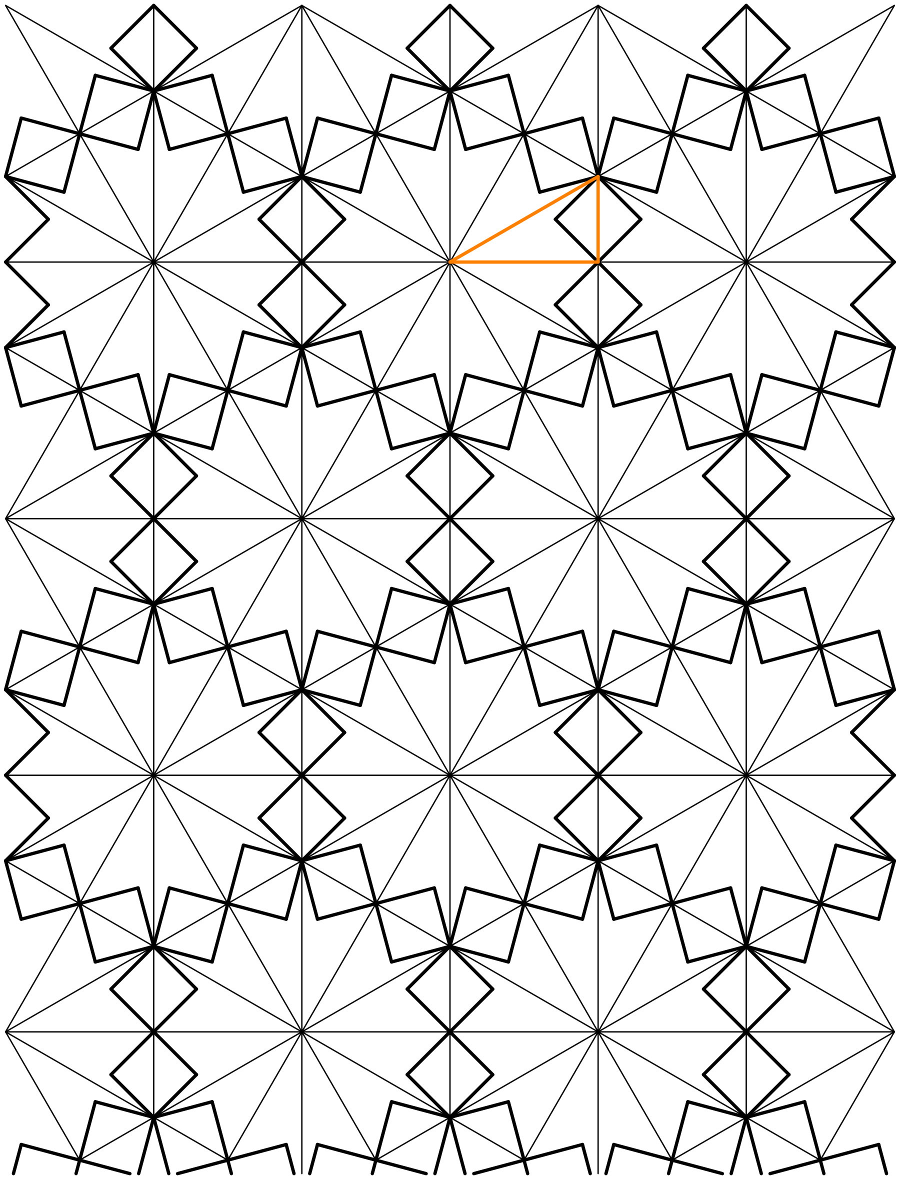

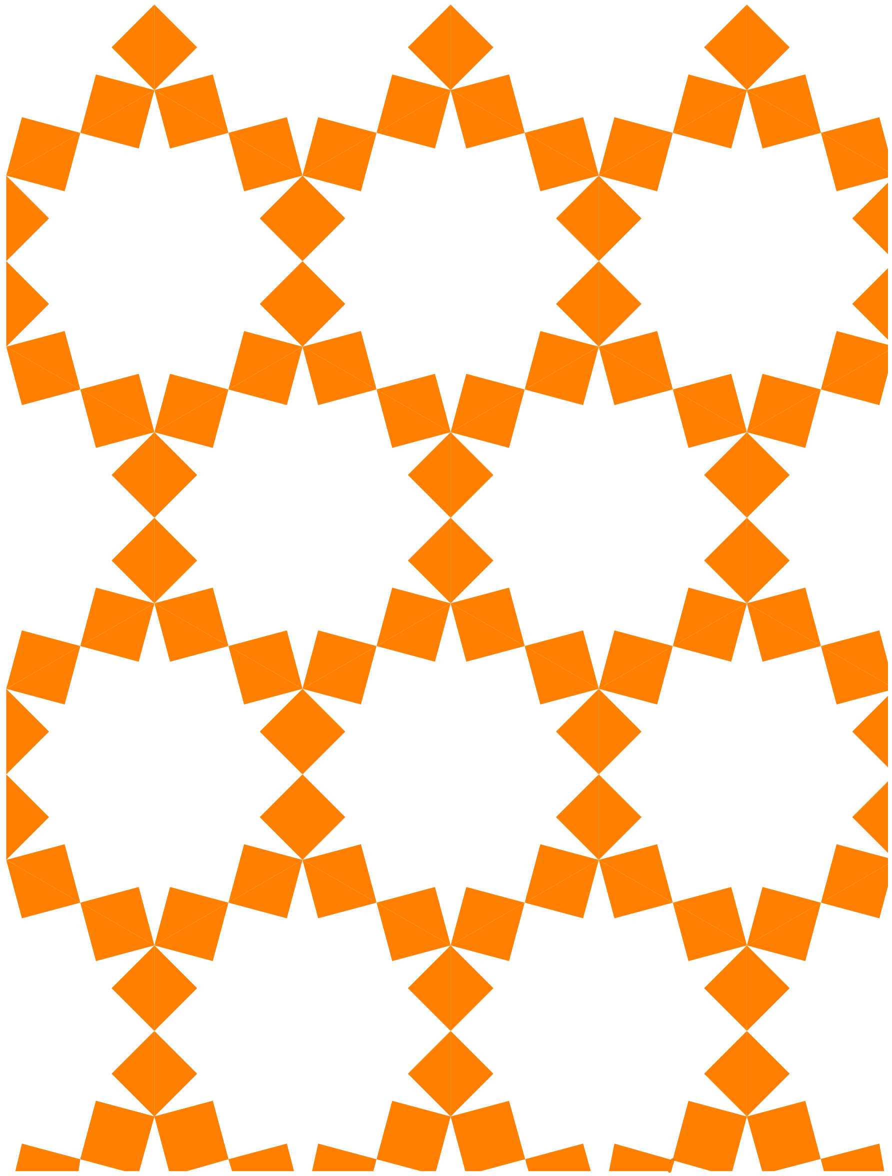

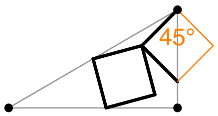
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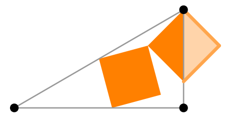
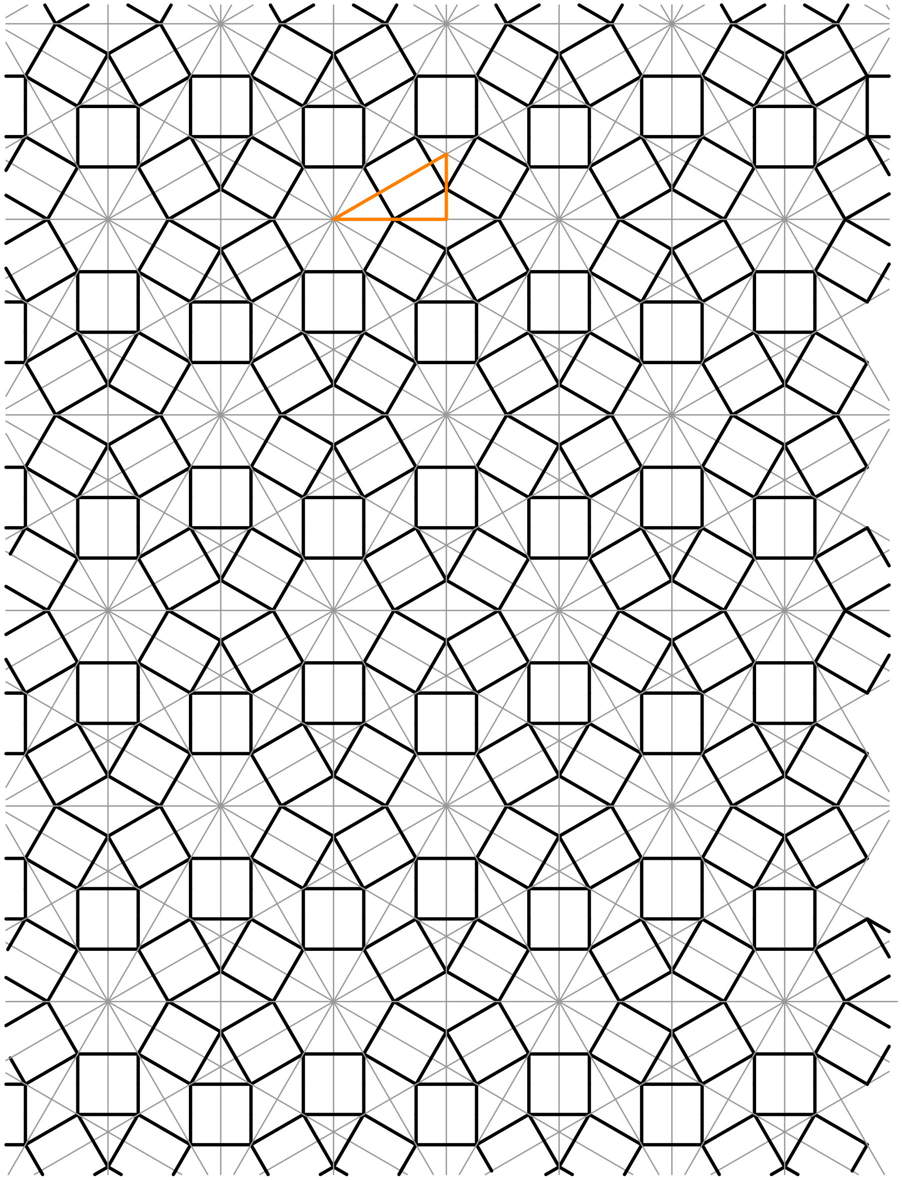
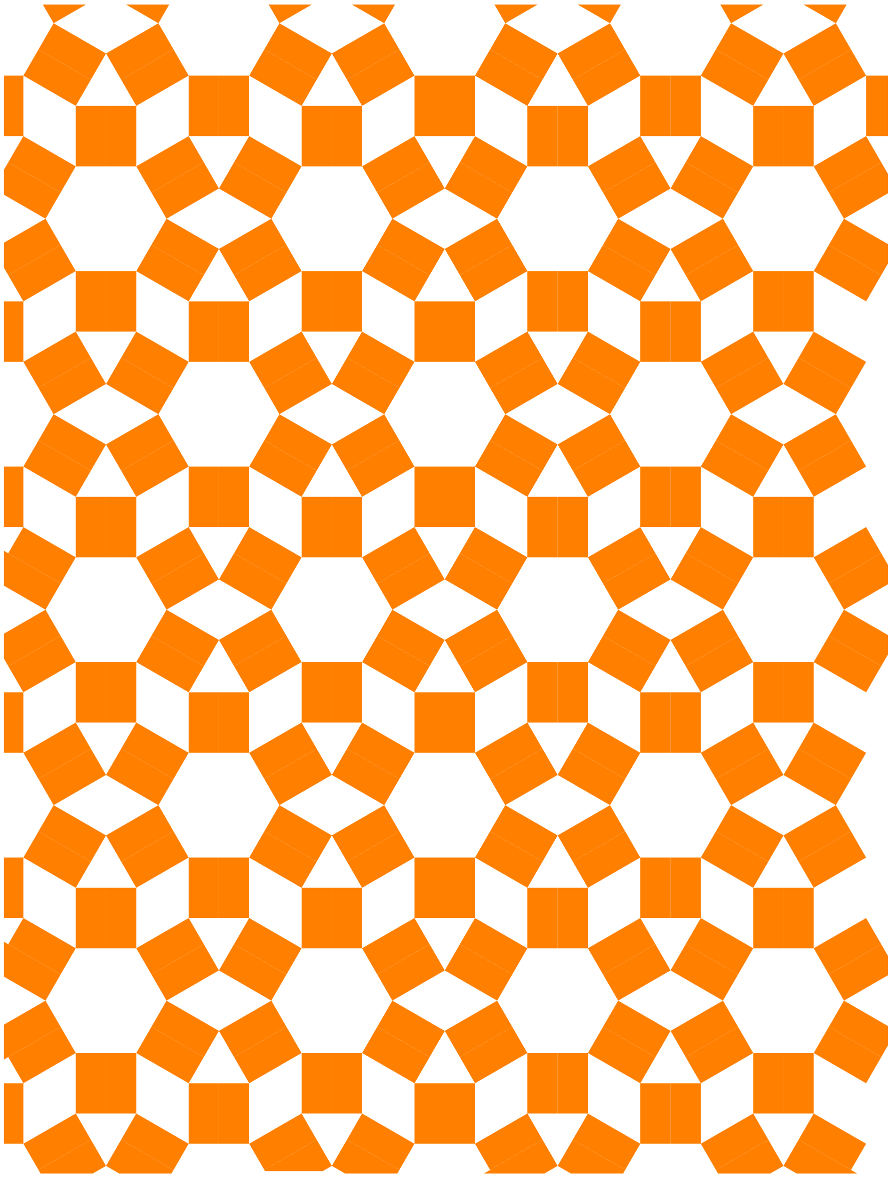
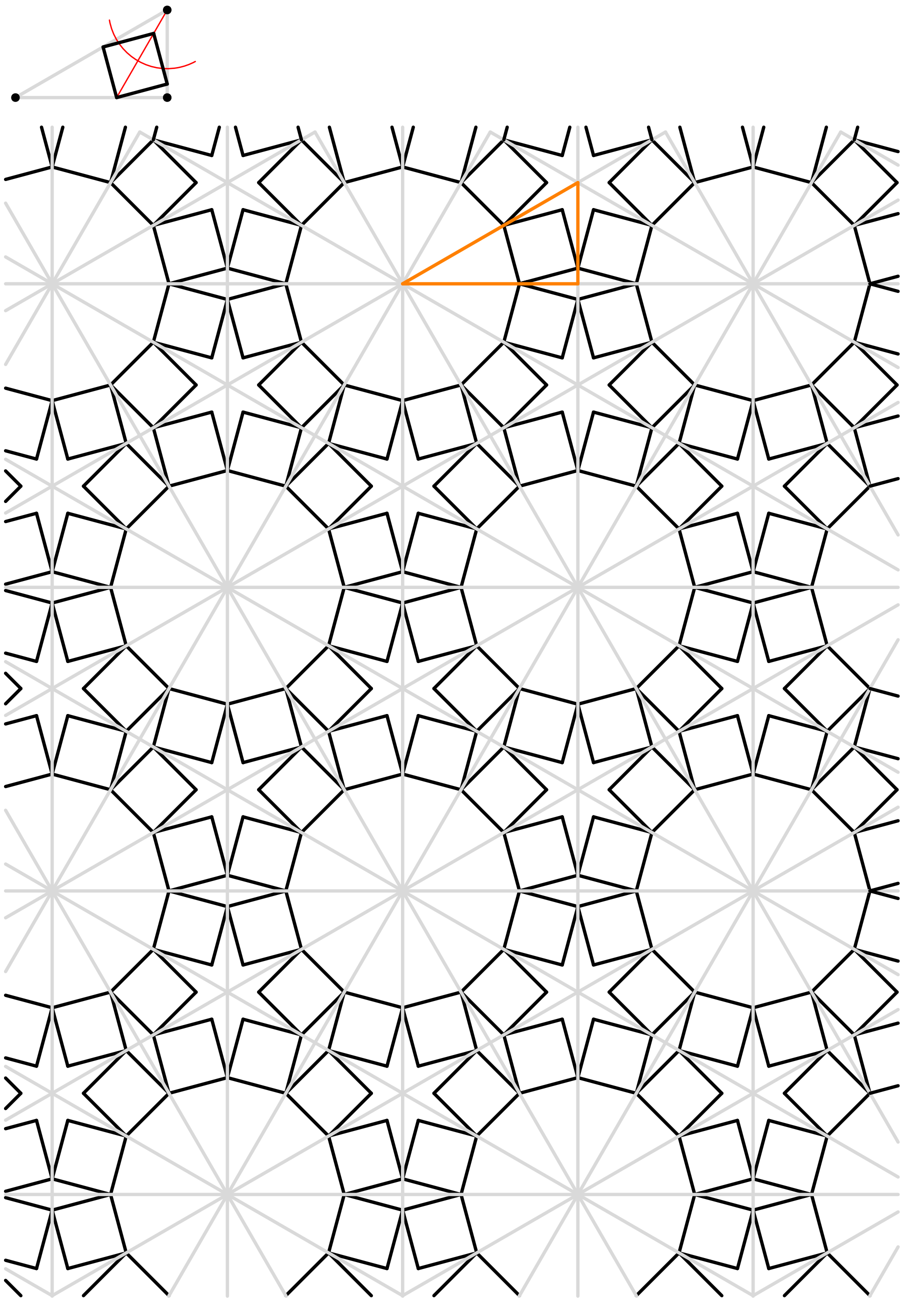
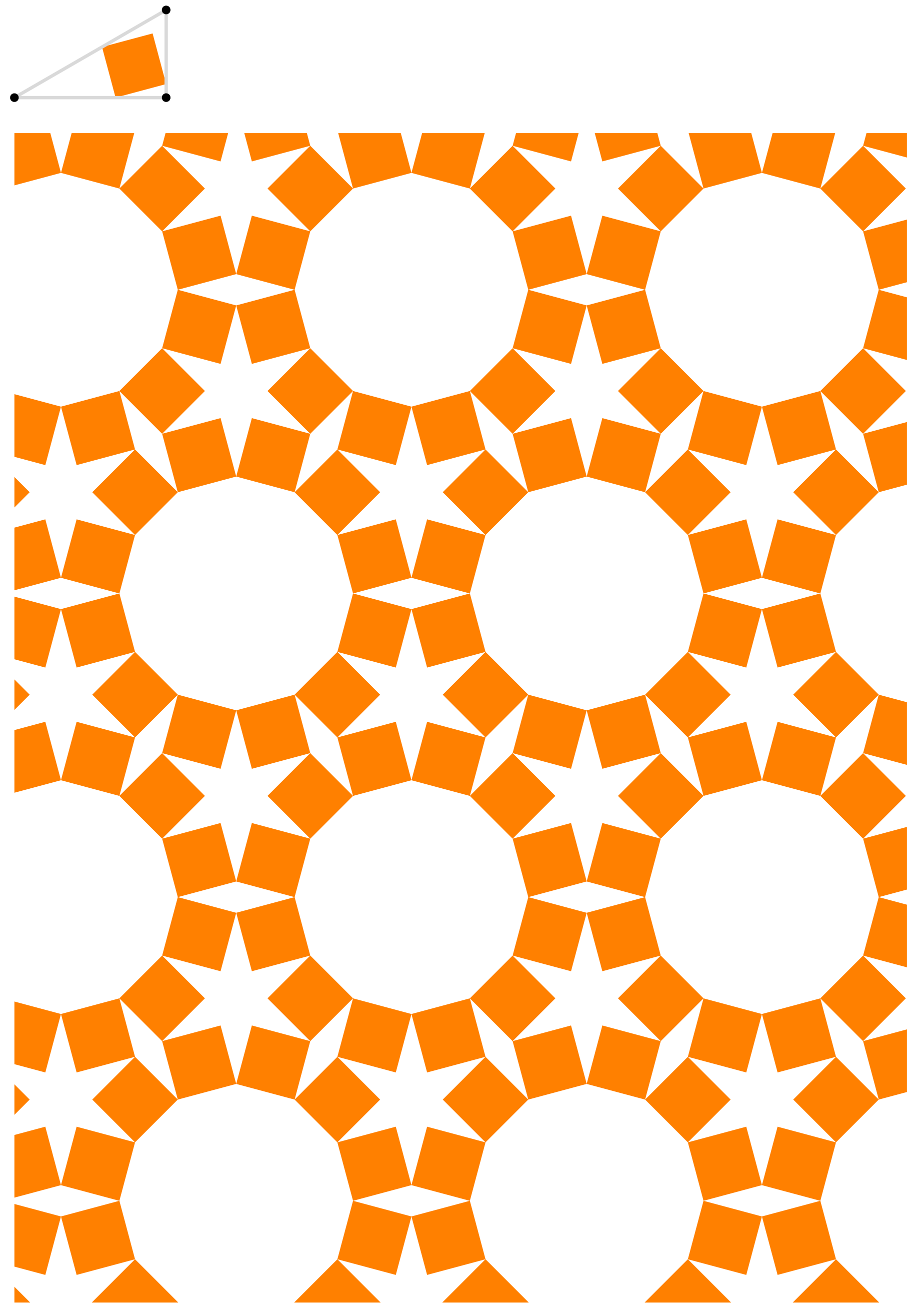




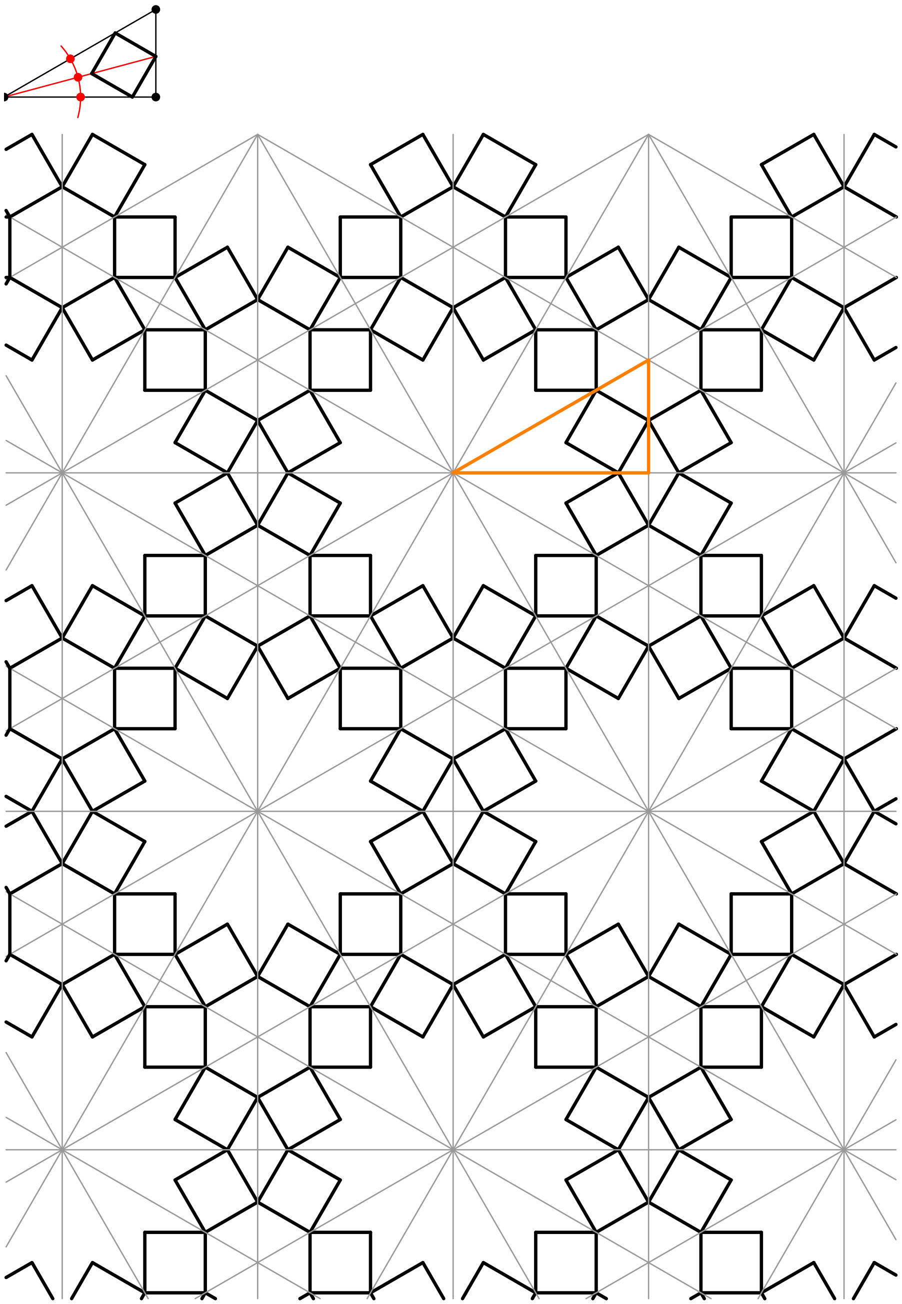


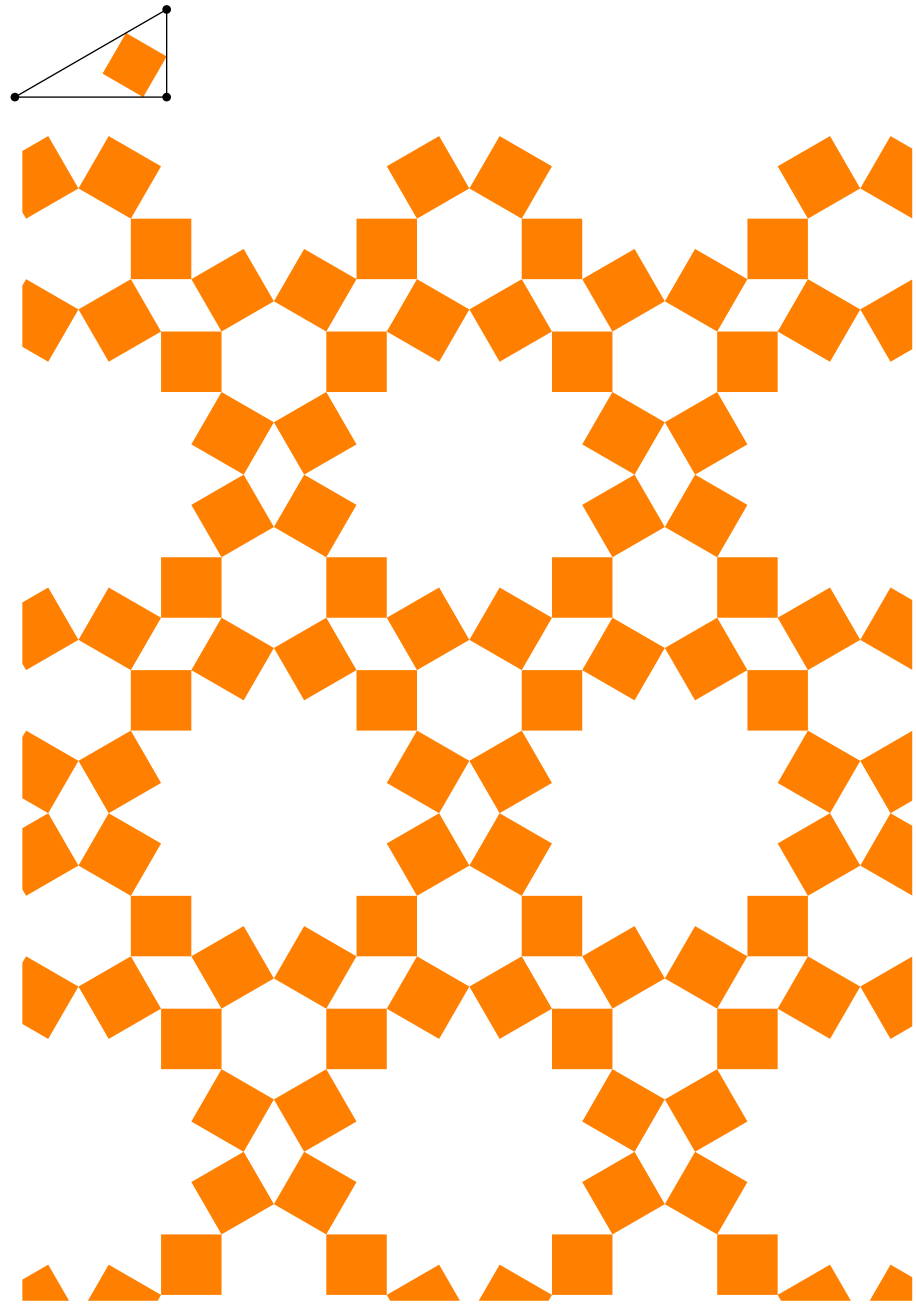
• B7 One square in a triangle • • Square Dance •



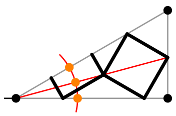


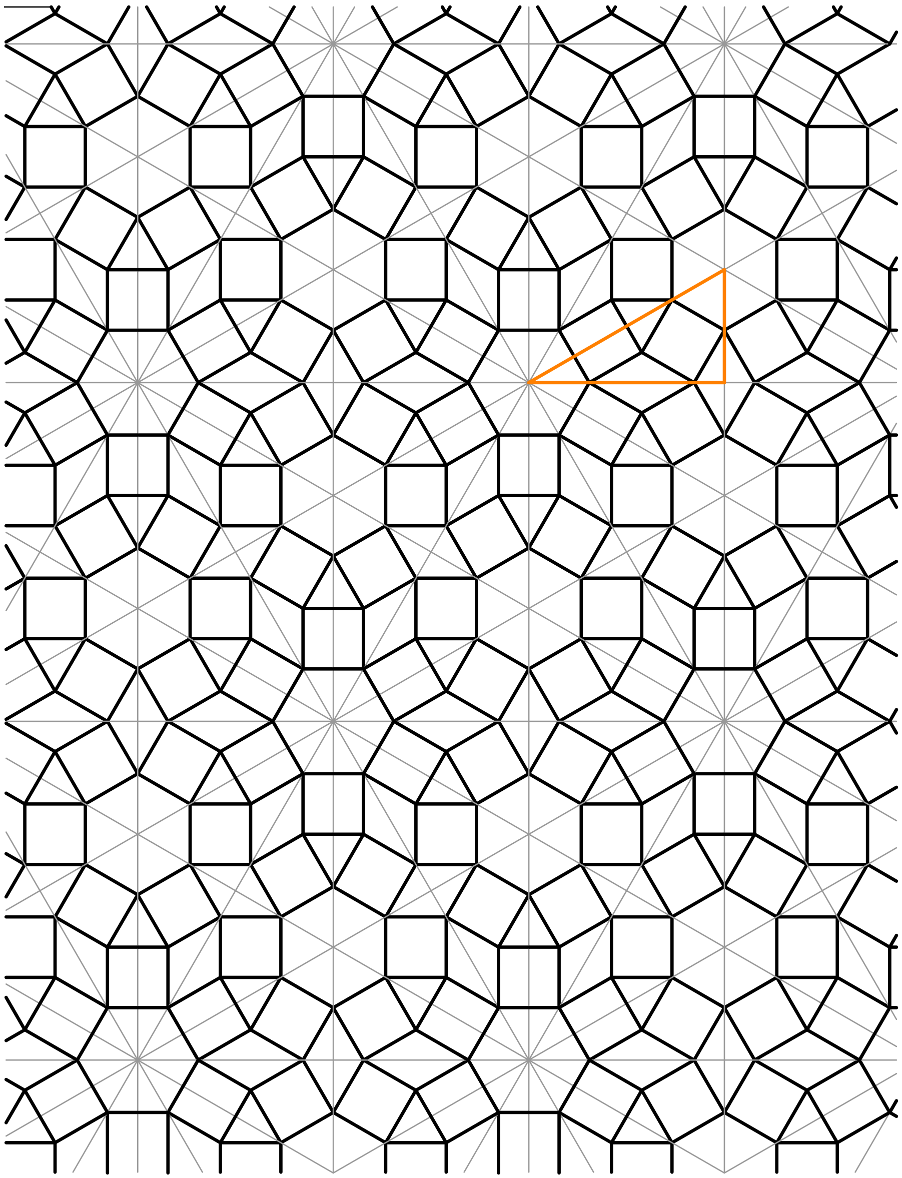
• B8 One square in a triangle •

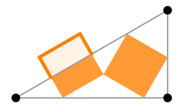


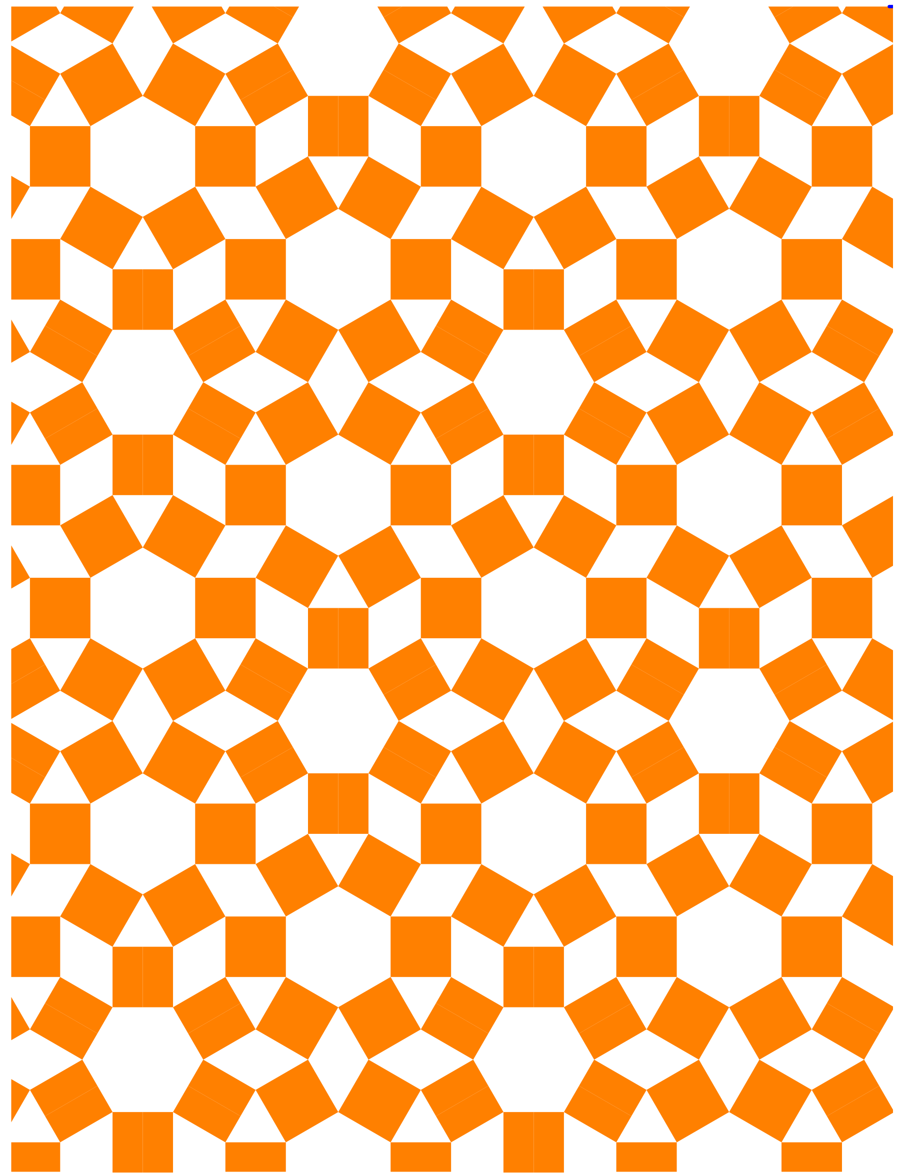


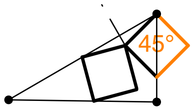
• B9 One and a half square in a triangle • • Square Dance •







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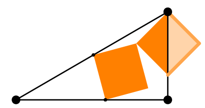
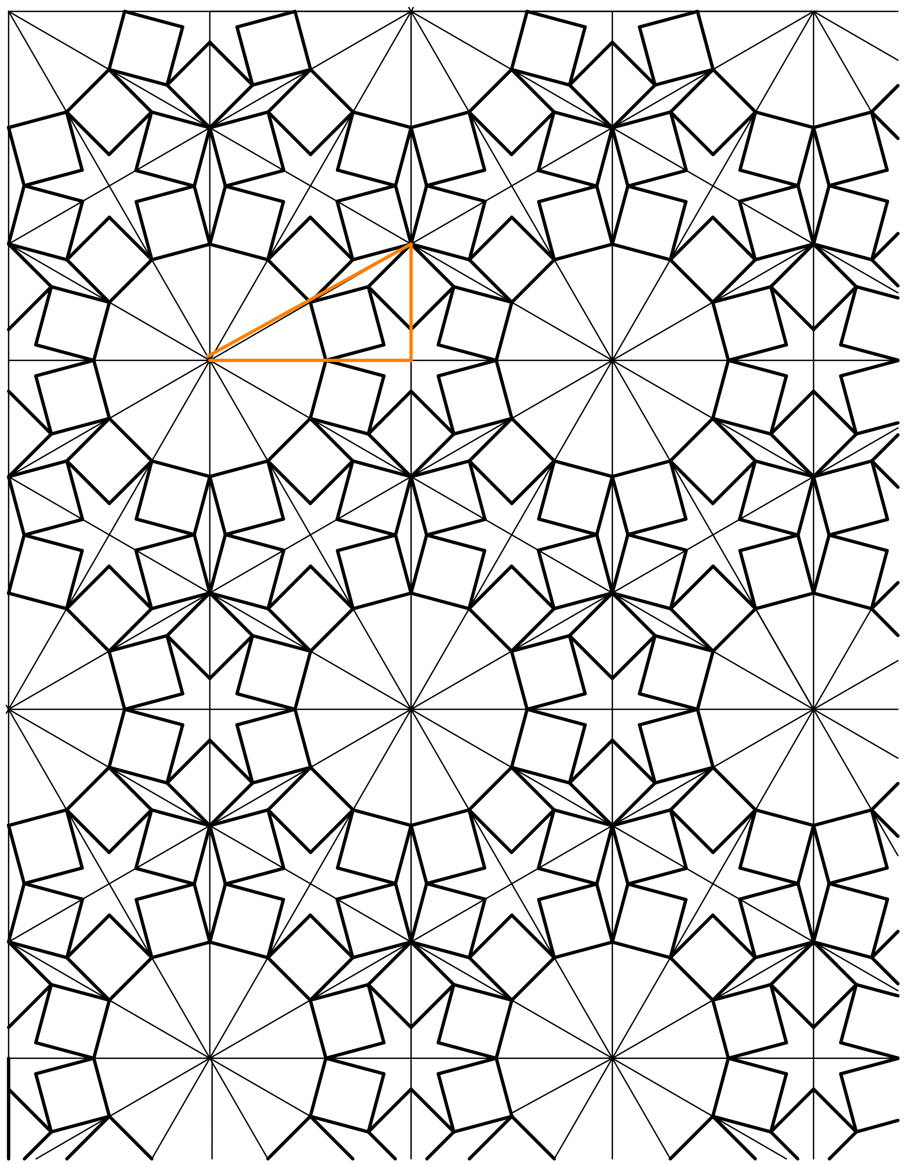
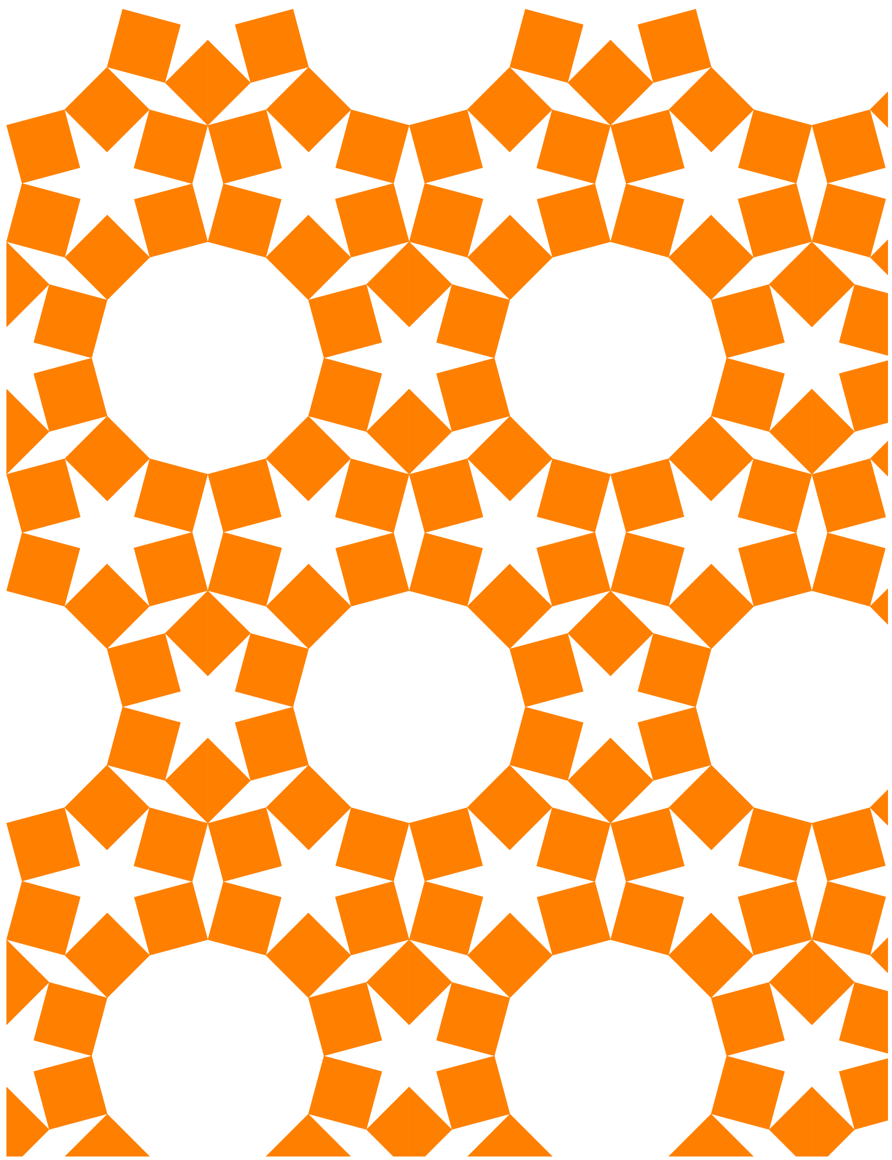
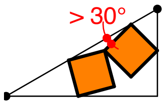
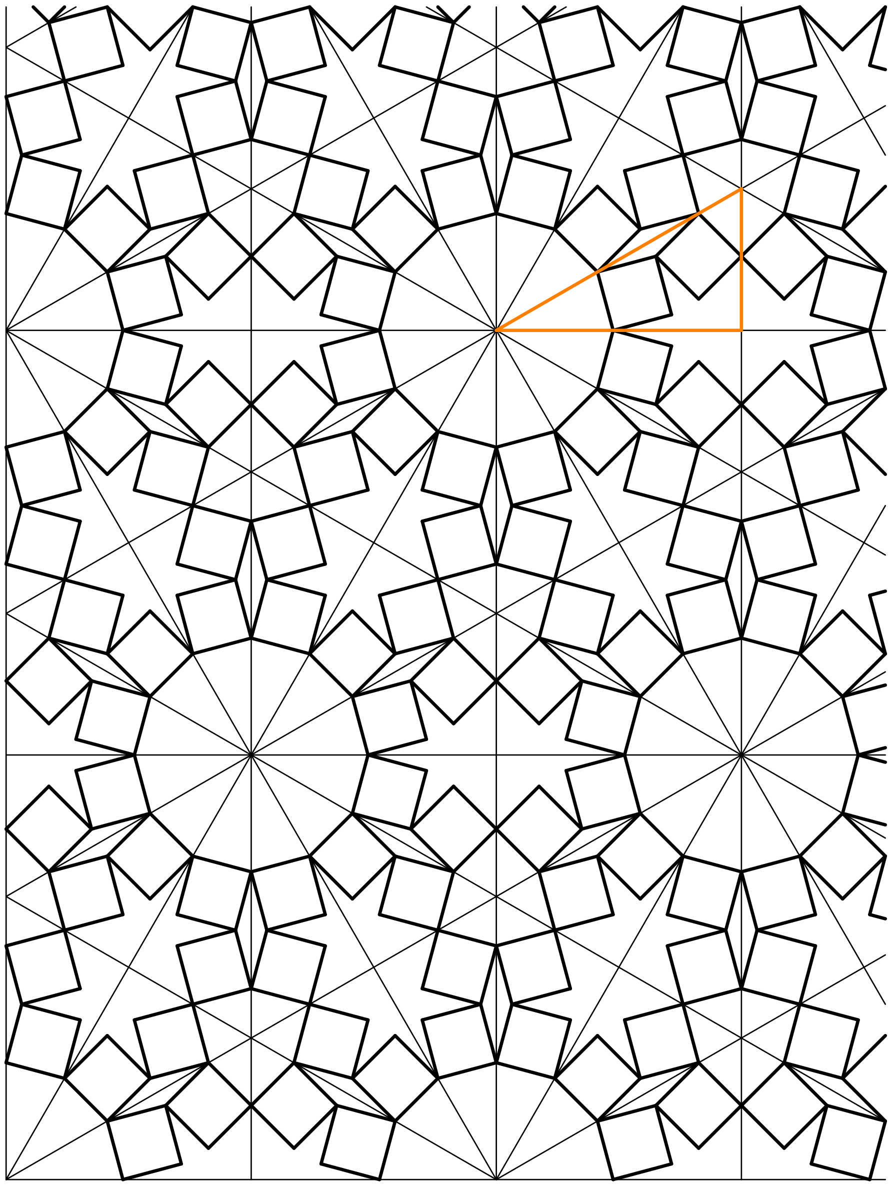
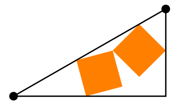


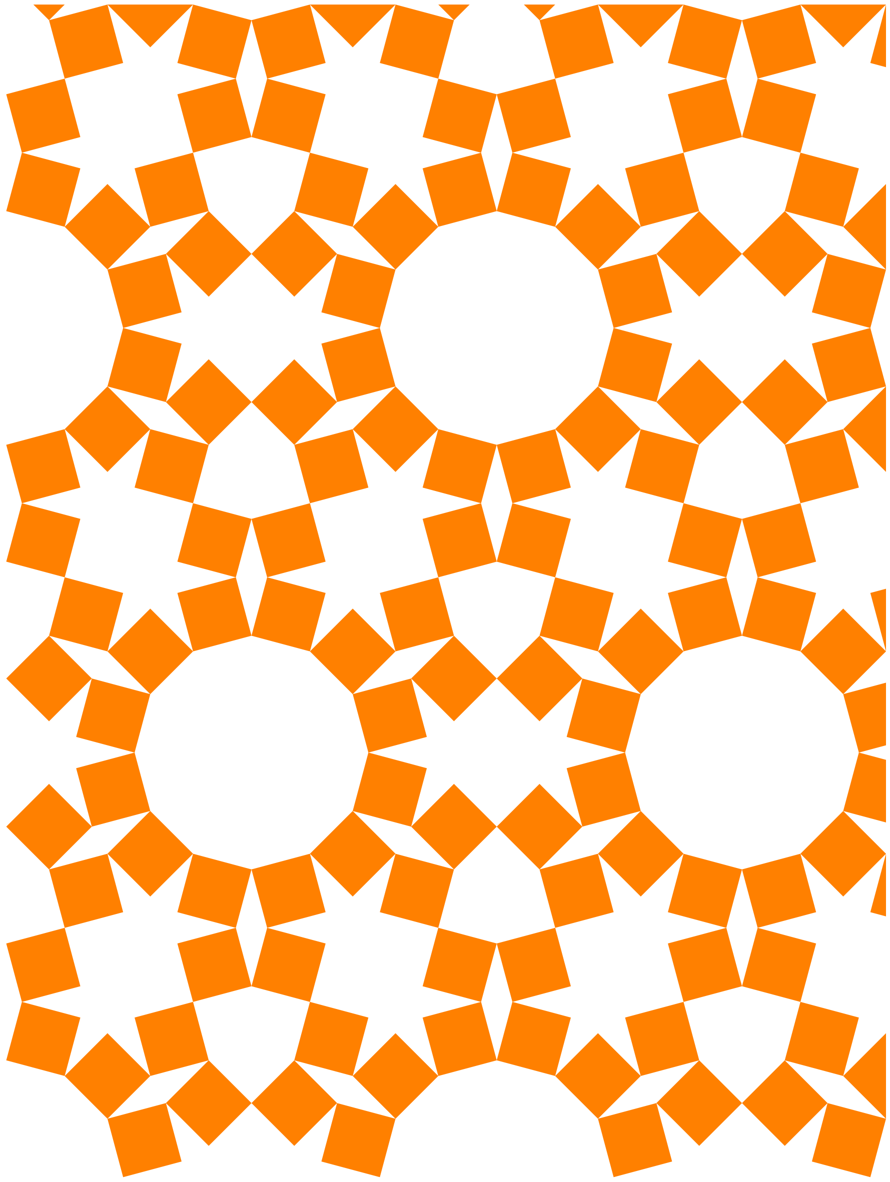


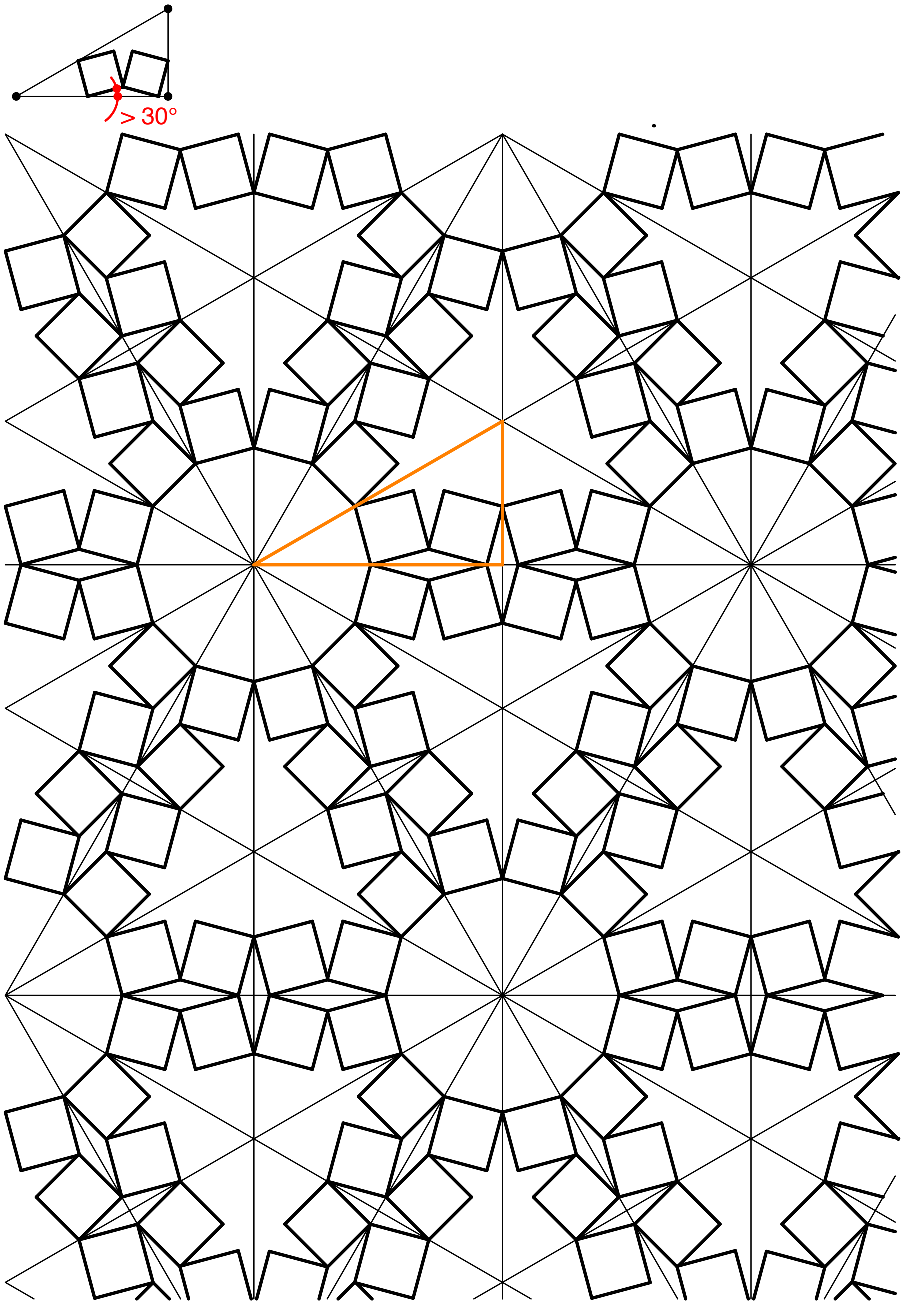


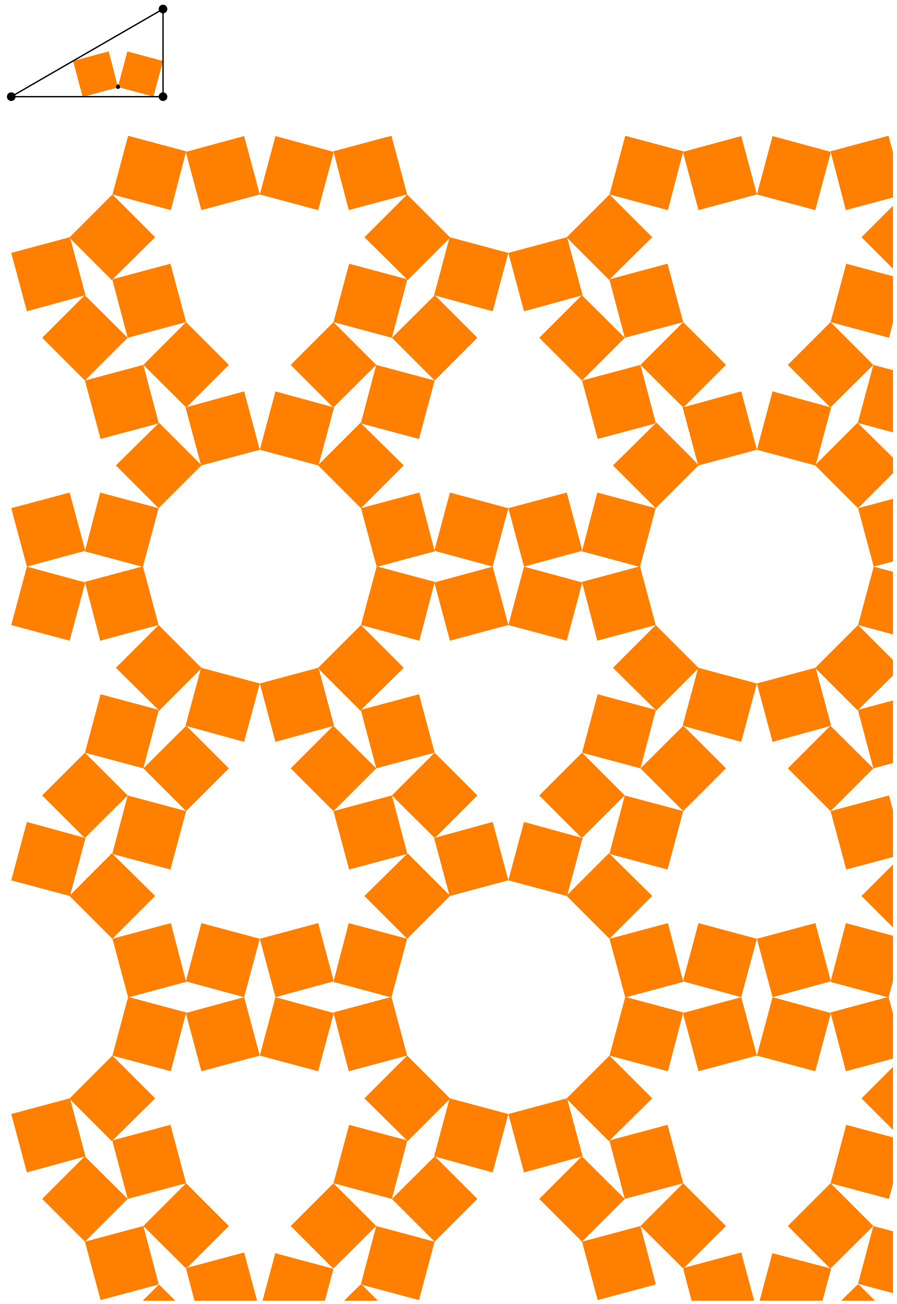

> 30°

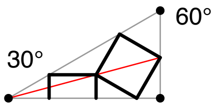




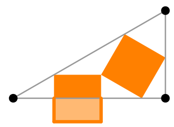


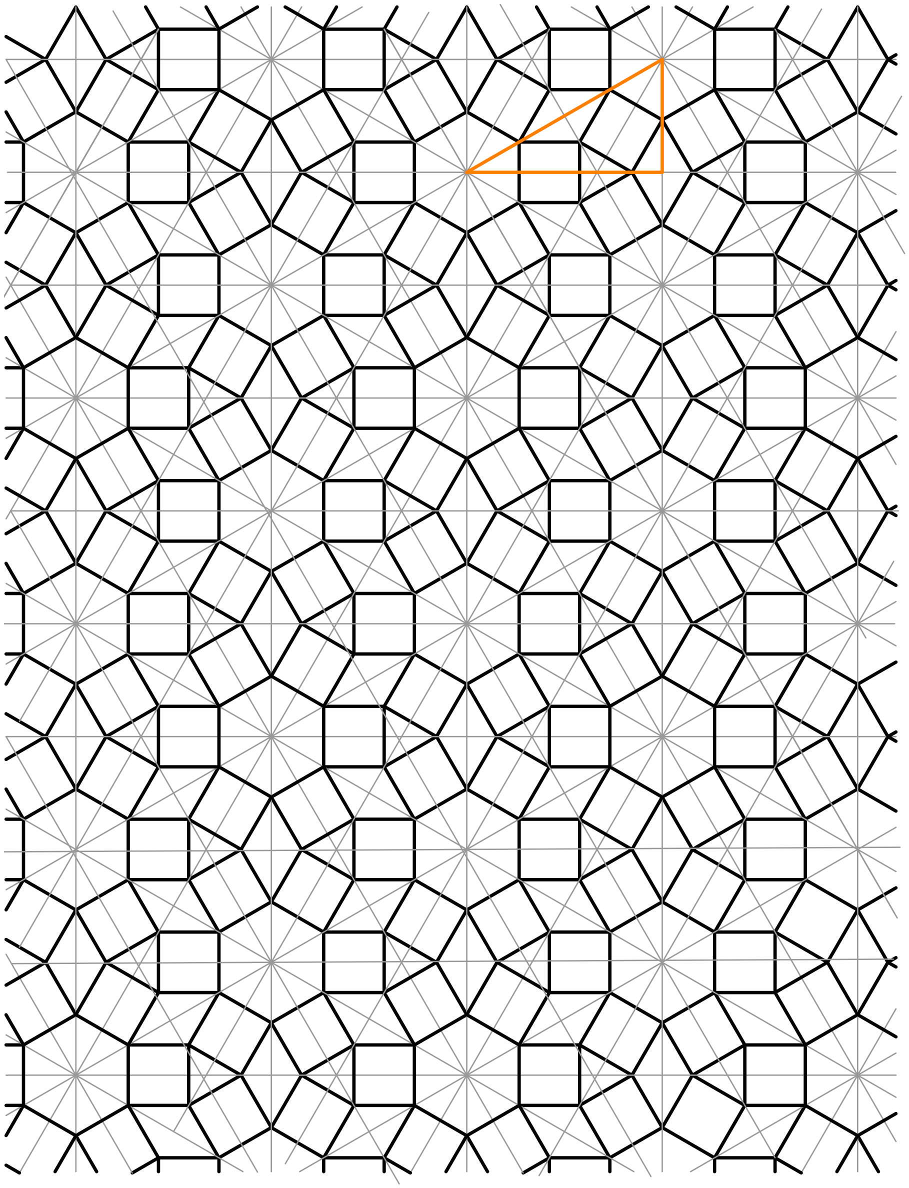

> 30°

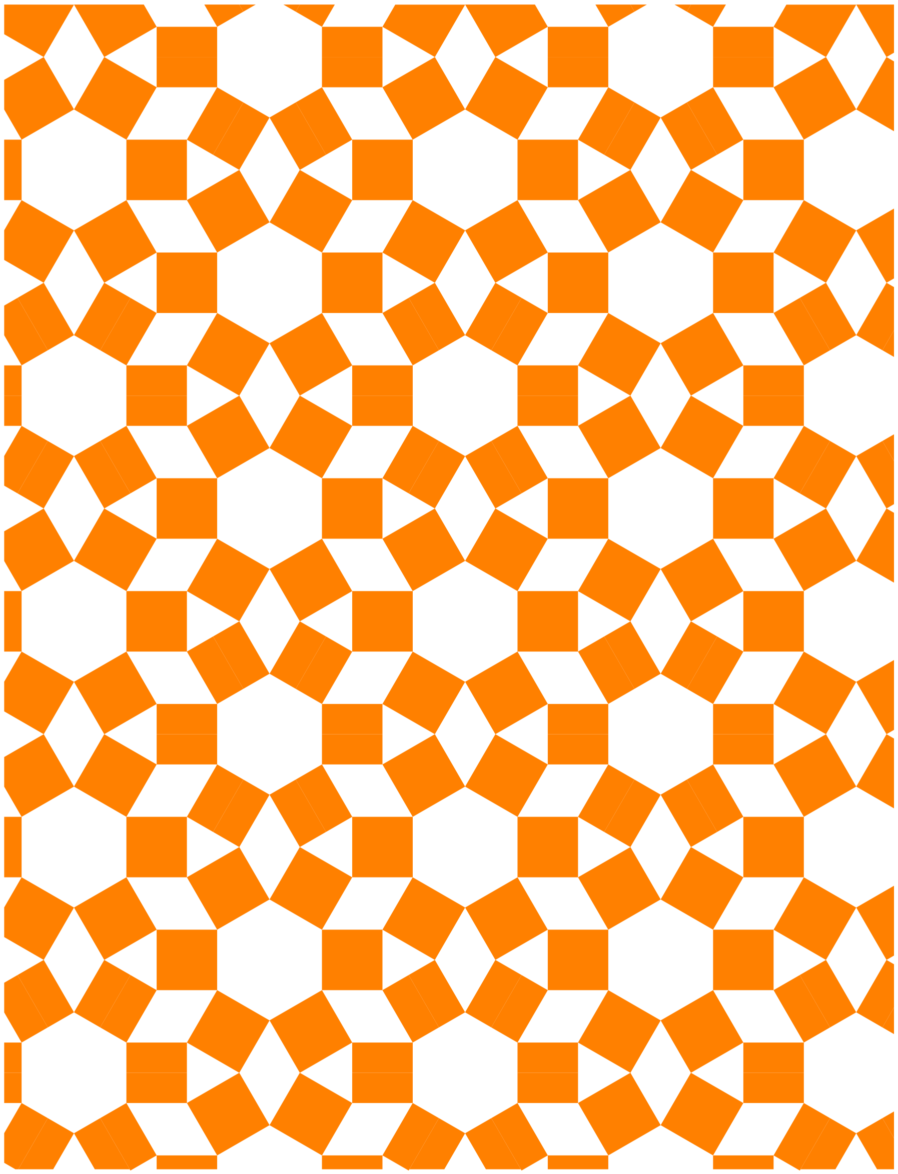




30°
60°

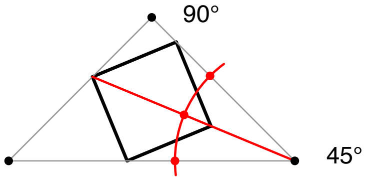


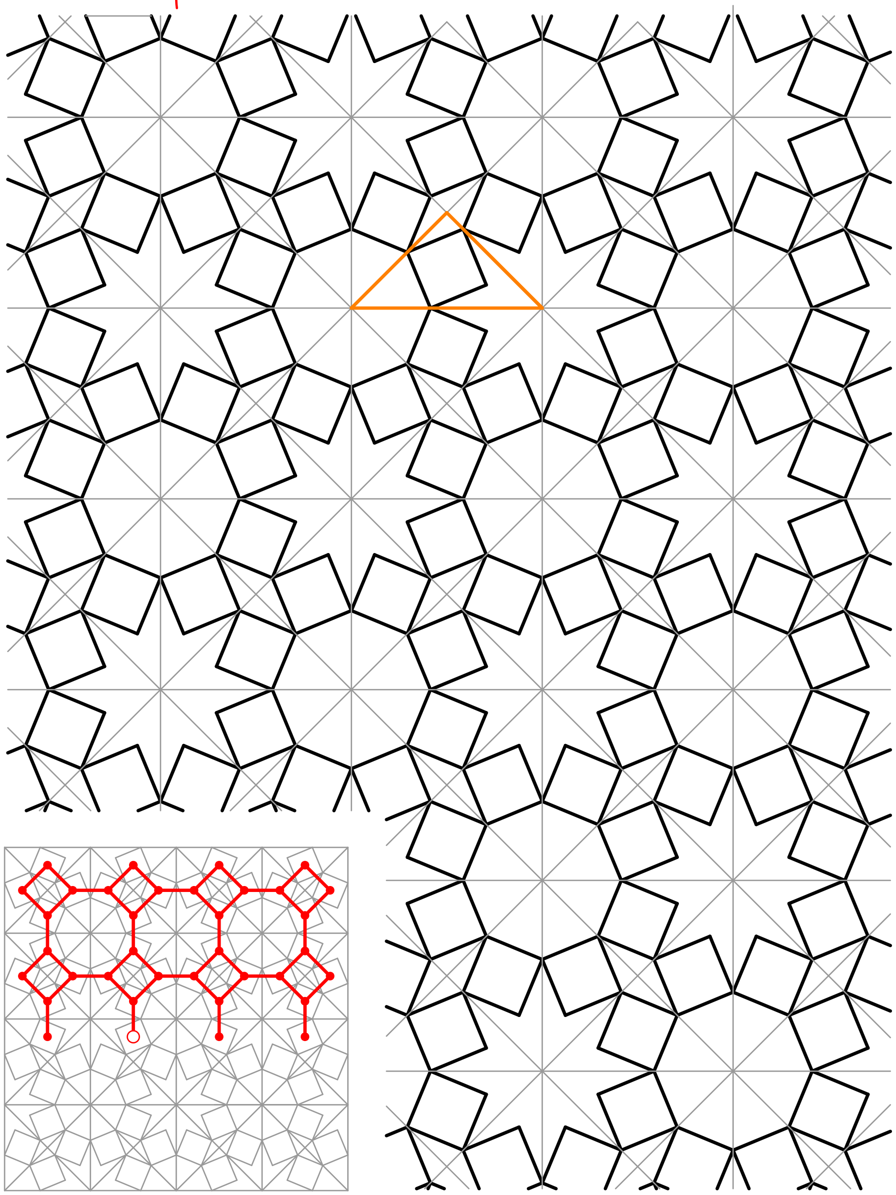


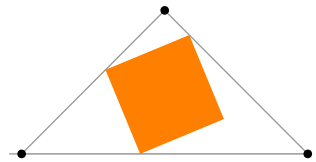


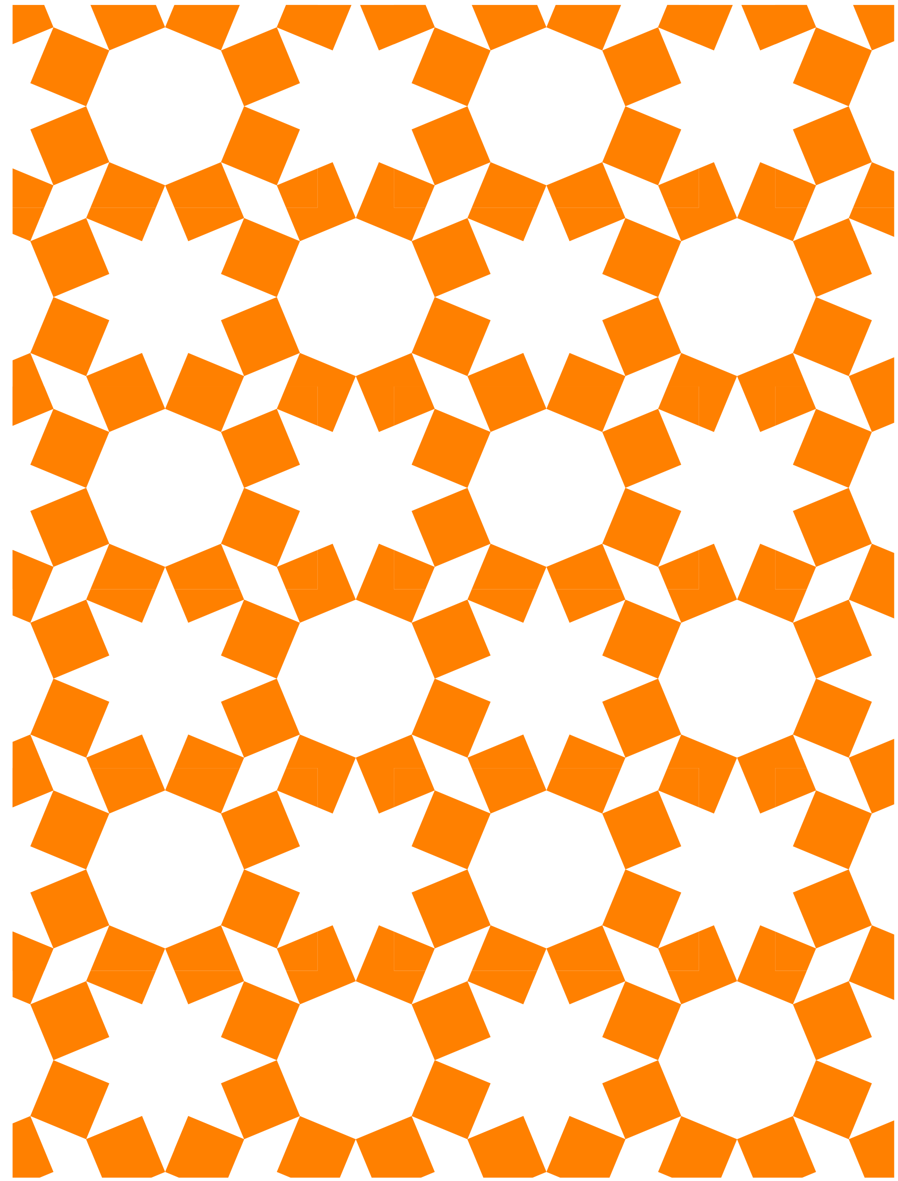
• B14 One square in a half square •

• Square Dance •

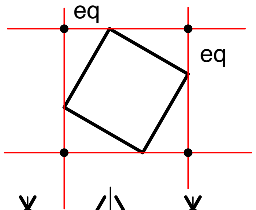


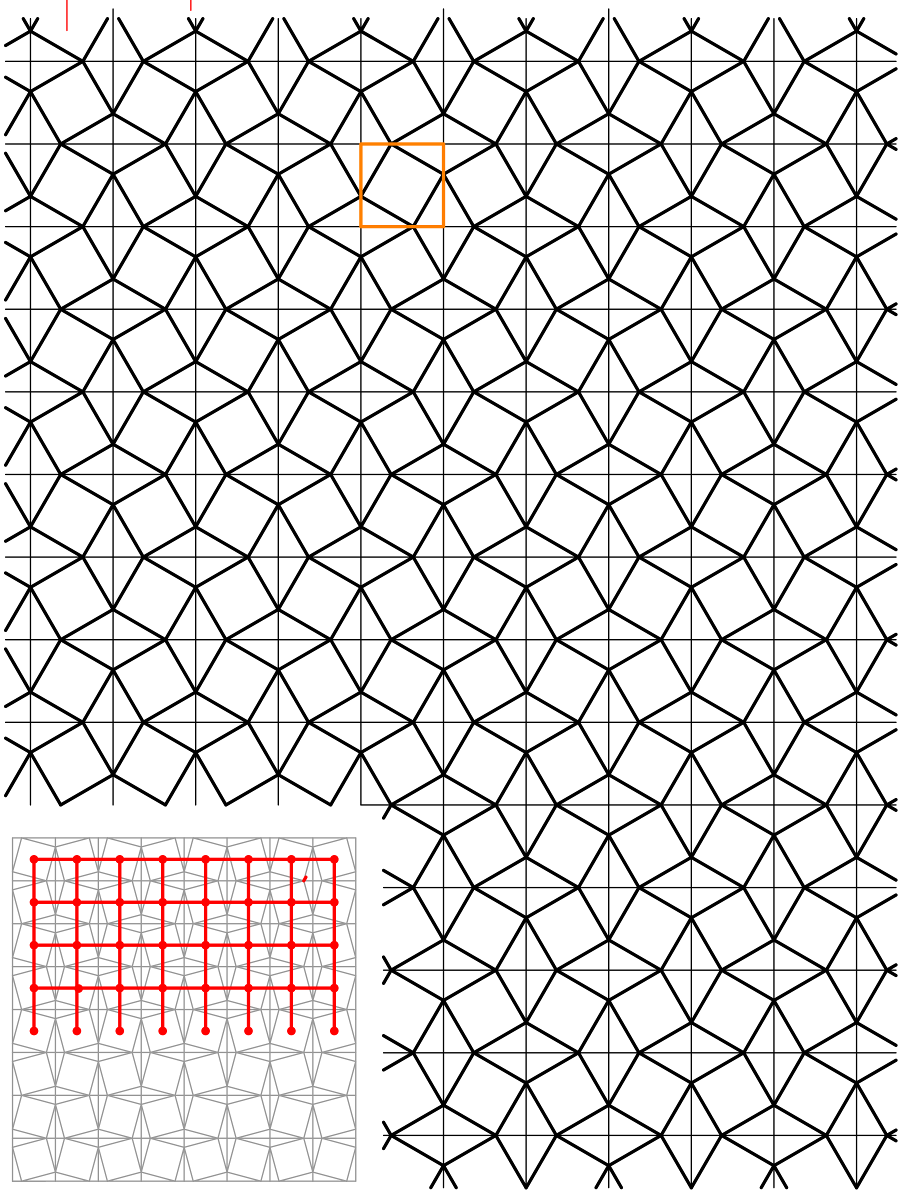


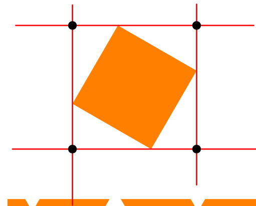


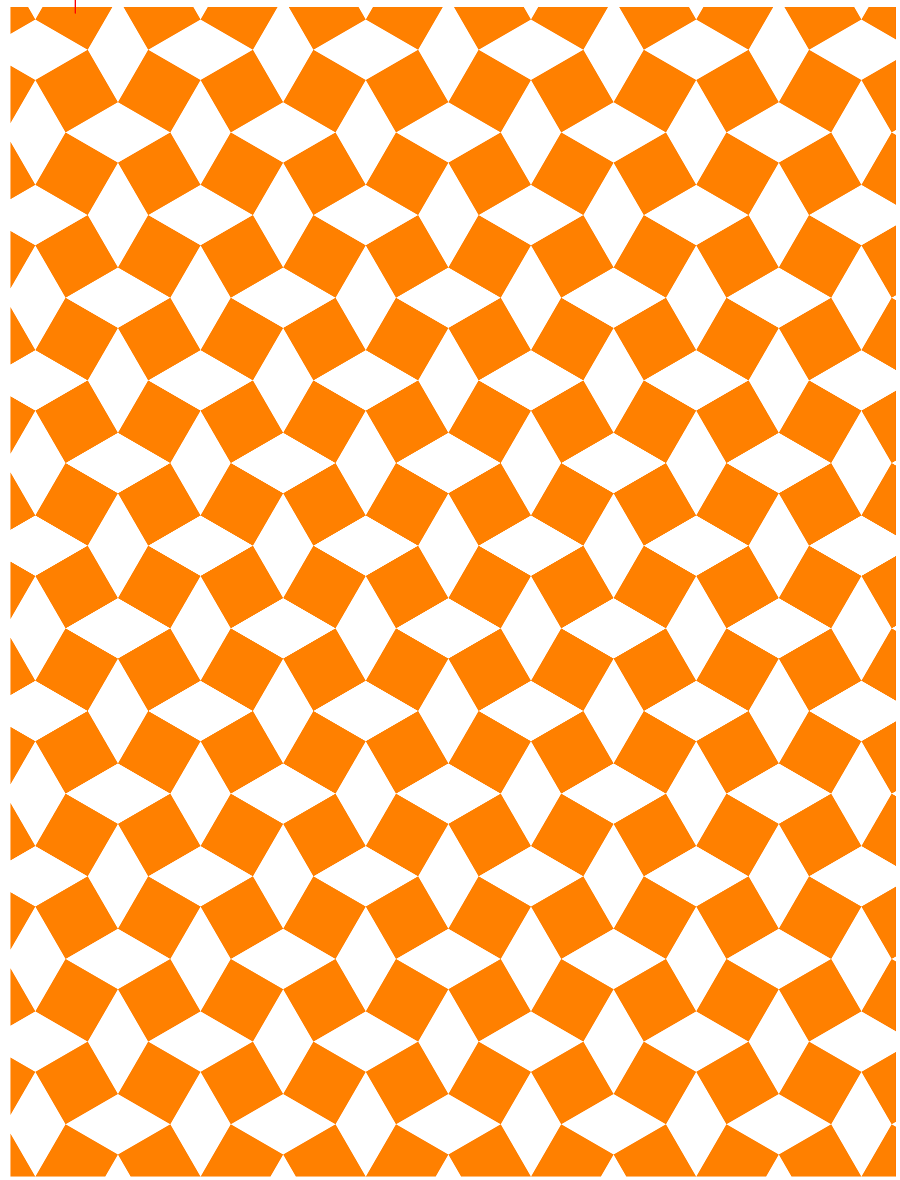


• B15 One square in a square •   • Square Dance •

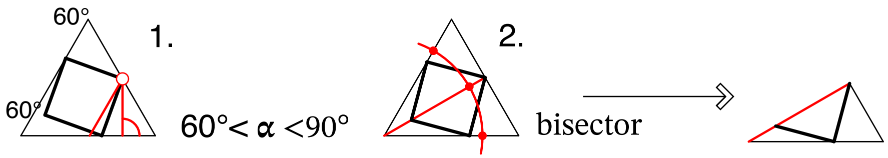


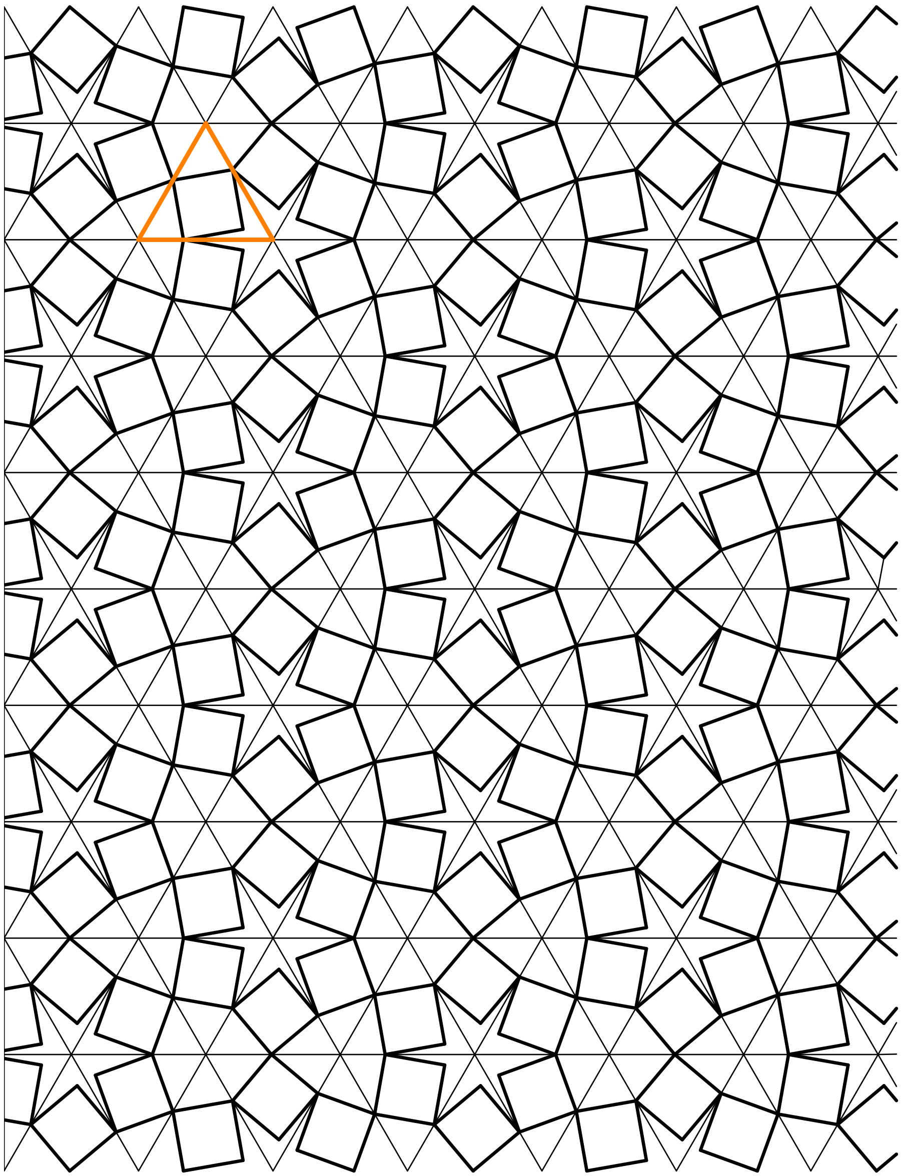






• B16 One square in e. triangle



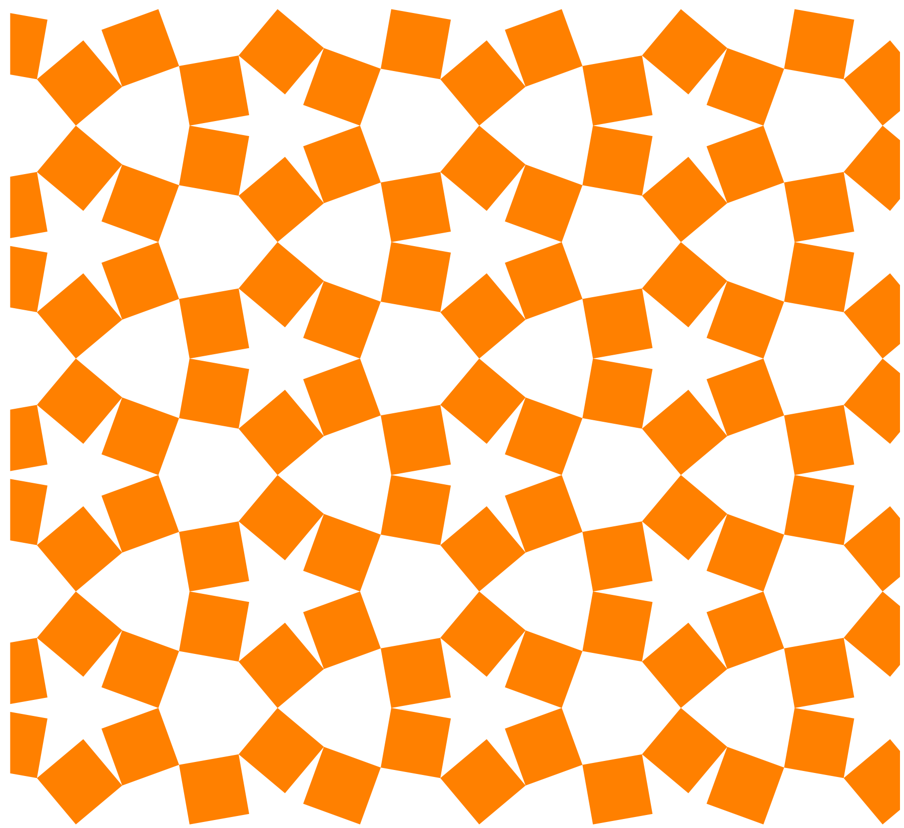


A square may be inscribed in an equilateral triangle in two different ways:

1. The square is drawn first, then the three edges are drawn passing by the vertices at 60° to each other. The condition shown (60°< α <90°) prevents two squares from having a common edge.

2. Here, we draw the triangle first. The square is installed with the help of a bisector. The third triangle is half of #2, which means it results in the same pattern as B4

The slight variation between the two constructions (not shown here) are visually hardly noticeable.



Among the regular and semi-regular polyhedra, only two follow our conditions for the square pattern. The cuboctta-hedron has 12 vertices, 24 edges and 14 faces, 6 of which are squares. The rhombicosidpdecahedron has 60 vertices, 120 edges and 62 faces, 30 of which are squares.
If you are interested to find more examples, the recommen-ded source: Zallgaller V. A. 1969 Convex Polyhedra with regular faces, Plenum Publ.CO.

Cuboctahedron
3.4.3.4

Rhombicosidodecahedron
3.4.5.4