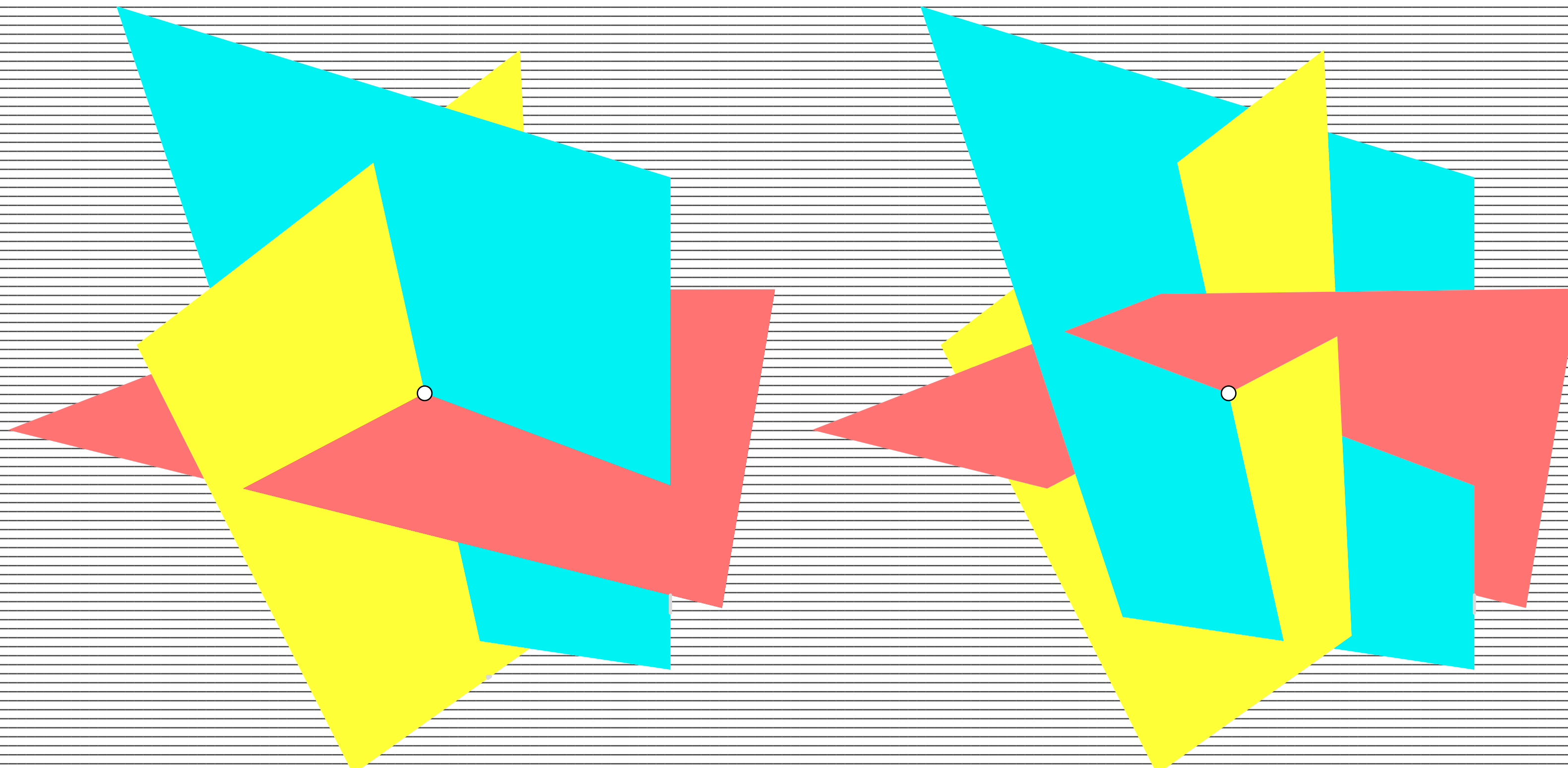
geometria2

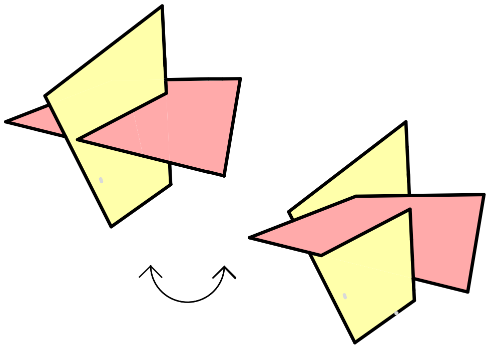
Intuitive 3-D Geometry



A short introduction to a simplified 3-D geometry.

Janos Baracs Montréal 2025

· Spatial Perception ·



The following pages were prepared to assist users of the Geometri2 software in acquiring basic skills for spatial geometry.

The question immediately arises: what is the use of learning somewhat complex geometric constructions in the digital era of computer-assisted 3-D design?

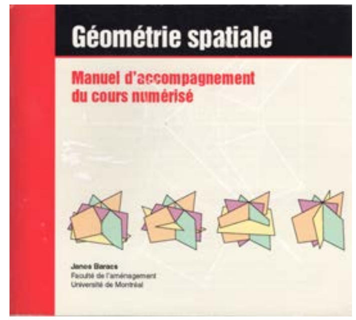
The knowledge of the geometric background puts the CAD user in charge of the computer rather than vice versa. When the computer is in charge of the designer, the design results clearly show unexpected and unexplainable solutions.

Acquiring a geometric background is the best way to develop and improve a crucial aptitude: spatial perception. Understanding a spatial relation and forming a mental image of a 3-D configuration is the best path to a creative process.

It is difficult and dangerous to predict the role of Artificial Intelligence in future design, but it is difficult to imagine that ignorance of geometry will be the answer.

Use the graphic software Geometria2 to practice problem-solving in spatial geometry. Enjoy!

· Teaching geometry ·



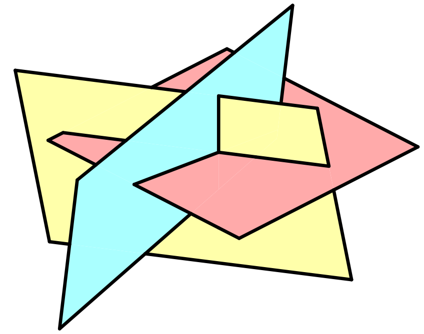
In the preceding, we made a case for the importance of geometric skills and aptitudes. We now make a few remarks about geometry teaching in our schools.

The modern design of objects and buildings indeed simplified the applied geometry. At the same time, in the last half-century, the teaching of 3-D geometry has been reduced or practically eliminated in North America. We don't know which came first, but we suggest that today's architects do not have the skills to draw or design a Gothic cathedral.

The most popular scientific method Descriptive Geometry was invented by French mathematician Gaspar Monge (1746-1818) to reproduce three-dimensional space in two dimensions. While it is still taught in high schools and universities in Europe and South America, it has disappeared recently from the curriculum in North America. Using 3-D software without 3-D geometrical skills is like writing poetry without knowledge of grammar.

3-D geometry is the most helpful tool for creatively manipulating space, and it should be taught again in design schools.

· Intuitive geometry·



Our method combines an axonometric projection of a spatial configuration with its projection on a horizontal plane.

This method (we named it Intuitive Geometry) can not fully replace Descriptive Geometry because it can not solve metric problems (perpendicular lines and planes, true sizes). However, it can be a stimulating tool in conception, a bridge between a free-hand sketch and a CAD drawing.

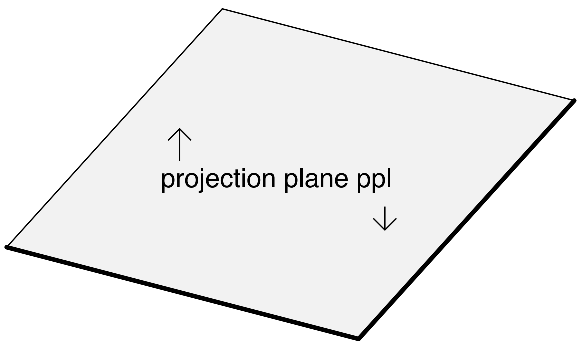
The following fifteen drawings present the basics of Intuitive Geometry:

Points (1)
Lines (2)
Planes (3)
Intersections (4, 5)
Problems (6, 7, 8)
Visibility (9)
Polyhedra (10-15)

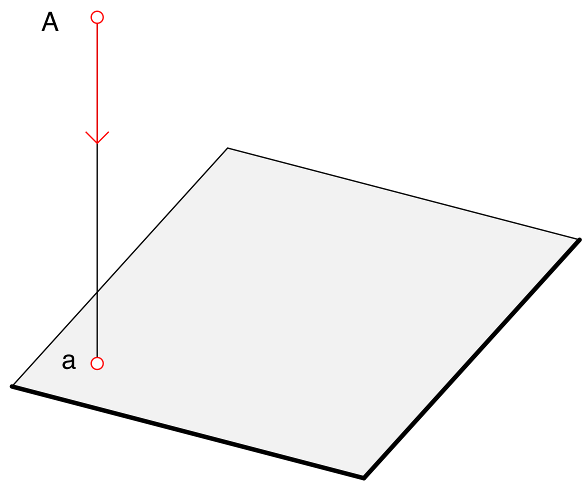
Given the limited space, we did not elaborate on whether a given problem has

1. Points

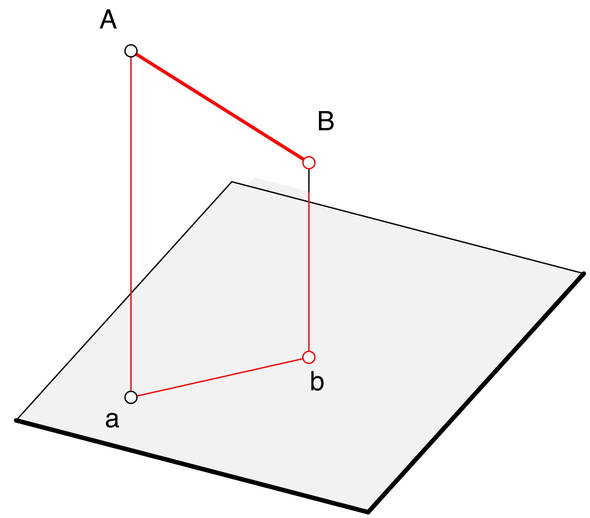
1.1 Oblique axonometry starts with a projection plane (ppl), the plane of the paper itself. The contours have no significance, but they help to visualize. We assume that ppl is horizontal



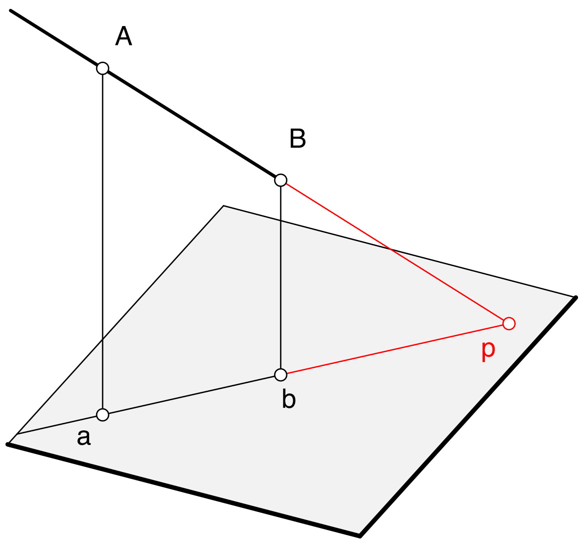
1.2 A point in OA is determined by its position in space A and its (oblique or orthogonal) projjection on ppl, a. The direction Aa is arbitrary; we assume that Aa is perpendicular to ppl, thus vertical.



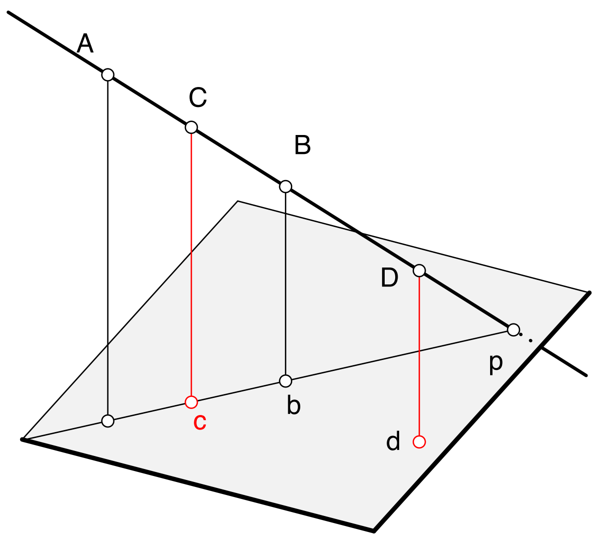
1.3 A segment AB is determined by the two points A and B.



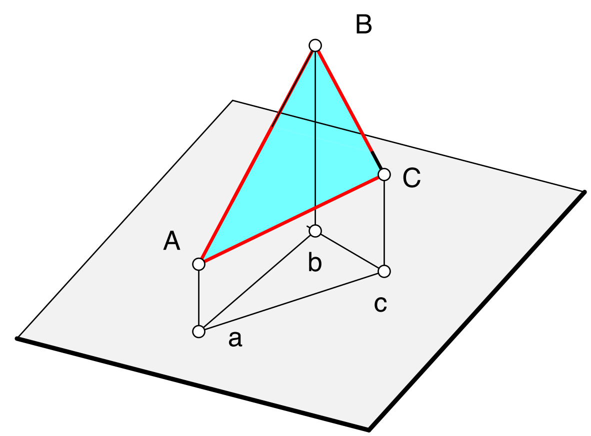
1.4 Given two points, A and B. Find the piercing point (intersection) p of the line AB and ppl.



1.5 Le point C is on the line AB (incident with), the point D is not on the line AB

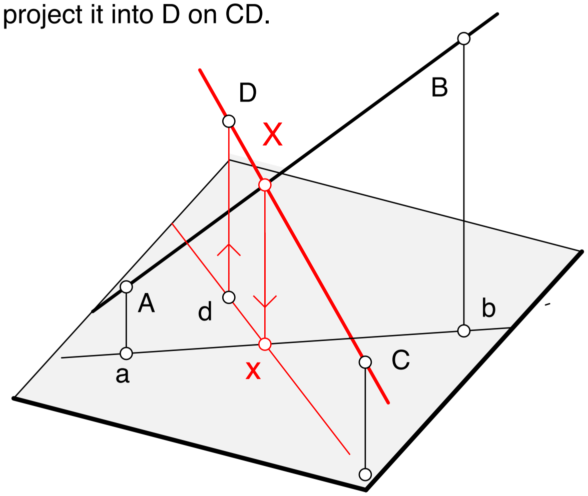


1.6 The plan of the triangle ABC is determined by the points A, B and C.



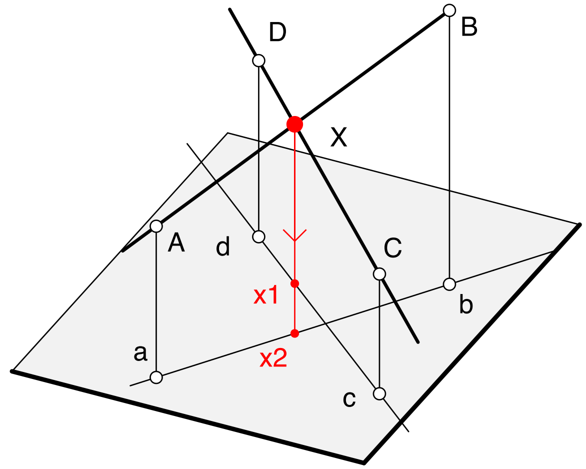
2.1 Given the line AB and the point C, not on the line AB. Construct a line CD, concurrent to the line AB.

Choose X on AB and project it onto x on ab. Join a and x, select point d on cx and project it into D on CD.



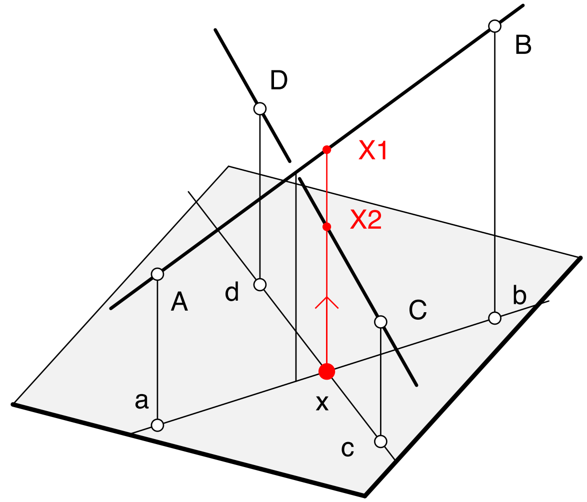
2.2 Given two lines, AB and CD. Find out if they are coplanar (concurrent) or not (skewed lines).

Project the “common” point X of the lines onto ppl; since X projects into two points x1 and x2, AB and CD are skew.



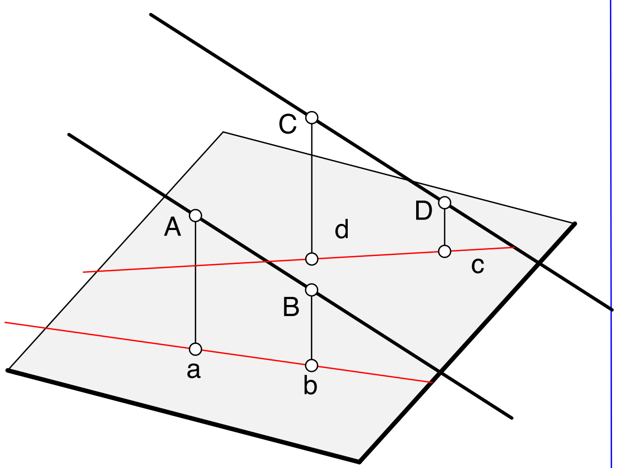
2.3 Given two lines, AB and CD. Find out which line is above the other one.

Project the “common” point x of the two projections onto AB and CD; since X1 is above X2 is above CD. See visibility!



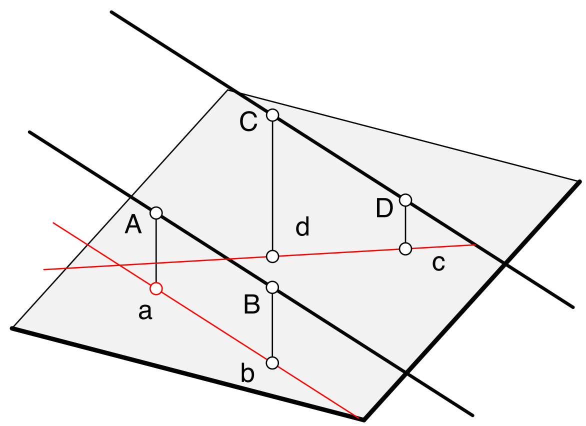
2.4 AB and CD appear to be parallel. They are not because their projections ab and cd are not parallel.

Theorem: the (parallel) projections of parallel lines on any plane are parallel.



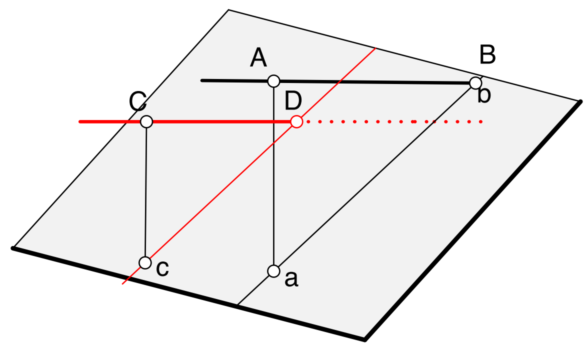
2.5 The line AB is parallel to the projection plane ppl because its projection ab is parallel to AB.

It follows that the line CD is not parallel to ppl, and AB is not parallel to CD.

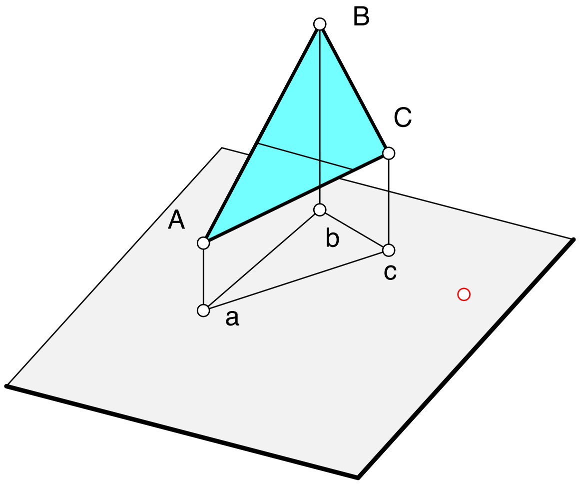


2.6 Given the line AB and point C. Construct the line CD parallel to line AB.

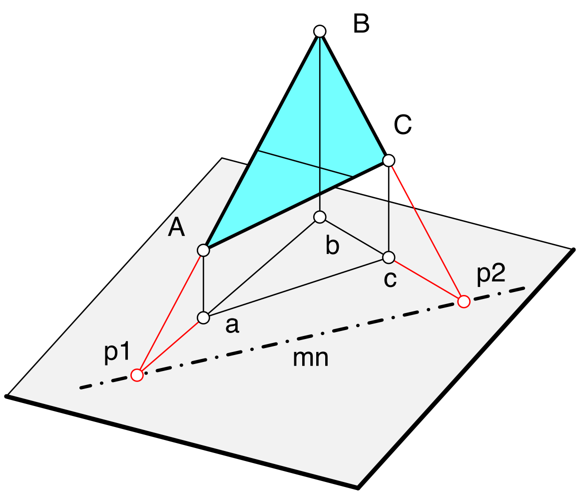
Draw parallels through points c to ab and through C to AB.



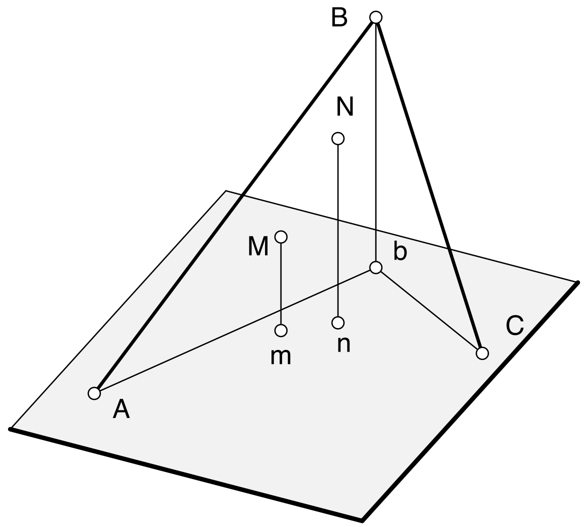
3.1 Given three points, A, B and C. Find the intersection mn of the planes ABC and ppl.



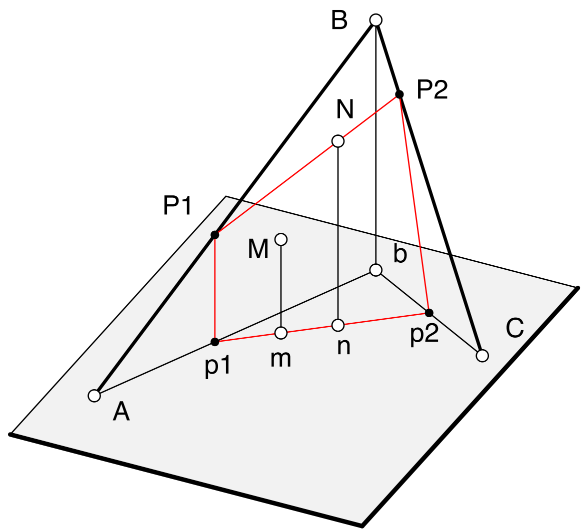
Construct the piercing points p1 and p2 with the plane ppl. The intersection mn is the join of the points p1 and p2.



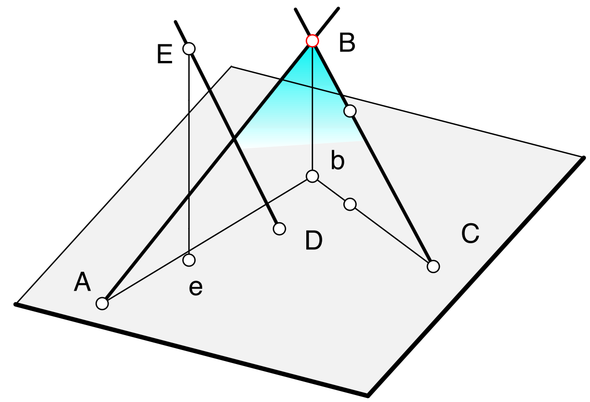
3.2 Given two lines, AB and BC, and points M and N. Determine if the points are on the plane.



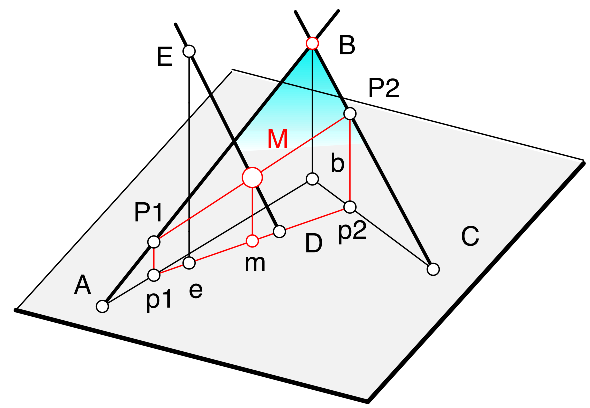
M is not on the plane, N is on the plane because N is on the line P1X2, which is on the plane.



3.3 Given the plane ABC and the line DE. Find the piercing point M of the line ED and the plane ABC.

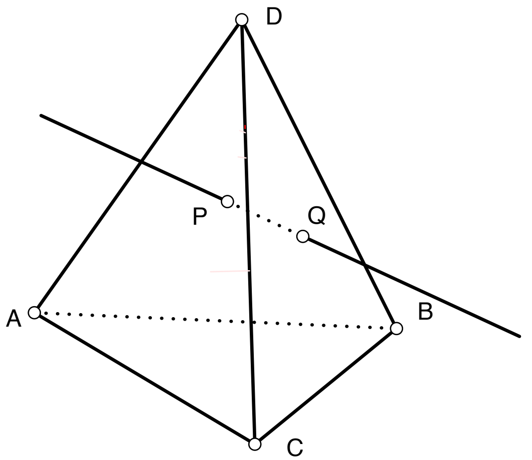


We pass a vertical plane through ED: join the points e and D. The line eD meets Ab in p1 and bC in p2. Project p1 into P1, P2 into P2. Join P1 and X2. The line P!P2 intersects ED in the common point M.



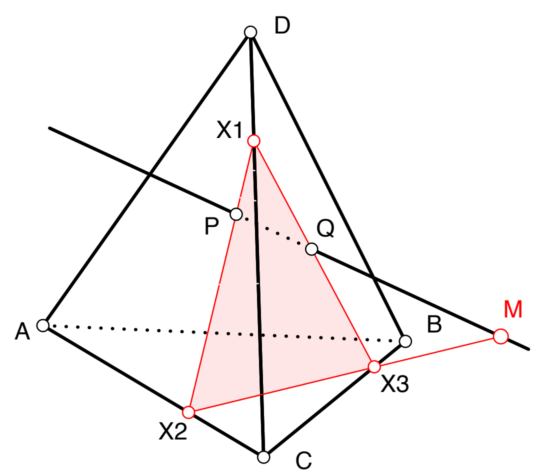
4.1 Given the tetrahedron ABCD and line PQ. Construct the piercing point M with the plane ABC. lines AB,

We no longer show the projection plane ppl: it plays no role in the constructions.

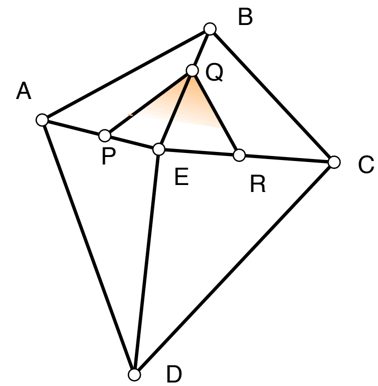


- choose X1 on DC
- PX1 meets AC at X2
- QX1 meets BC at X3
- X2X3 meets PQ at M

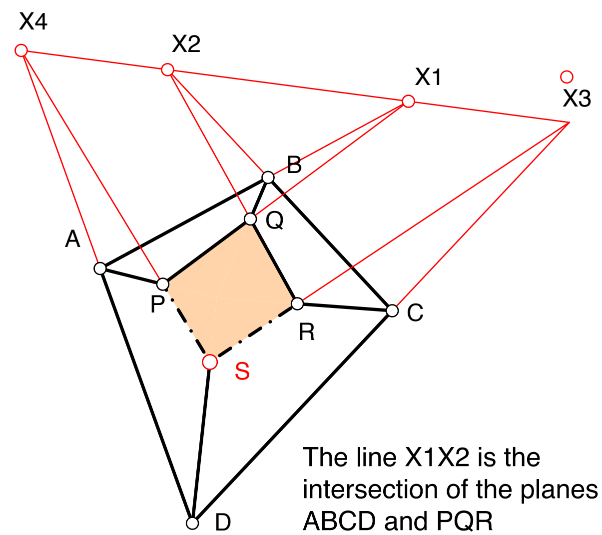
The auxiliary plane X1X2X3 meets the plane ABC in the line X2X3. The line X2X3 is coplanar with PQ.



4.2 Given the pyramide ABCDE and the plane determined by the points P, Q and R. Truncate the pyramid with the plane PQR: find the point S on the line DE.



- PQ and AB meet at the point X1
- QR and BC meet at the point X2
- DC and the line X1X2 meet at X3
- AD and line line X1X2 meet at X4
- the line PX$4 and RX3 meet at S



The line X1X2 is the intersection of the planes ABCD and PQR

4.3 Draw a triangle BCD with vertices on the lines AB, AC, and AD (Fig. 1.). The difficulty is shown in Fig.2. when an arbitrary starting point M is chosen on AC, the third line does not meet AC at M.

Fig. 1

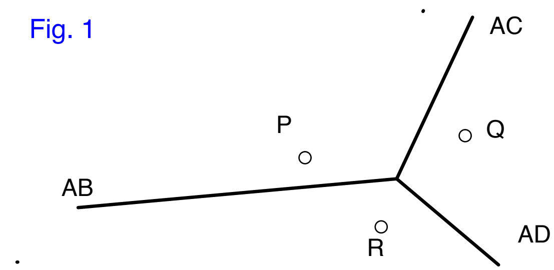
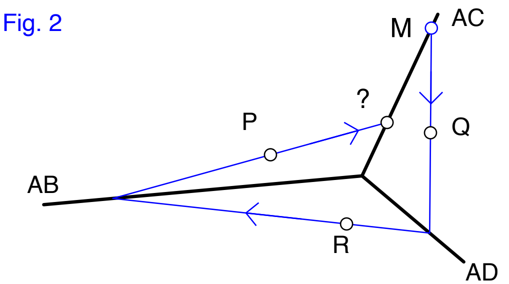
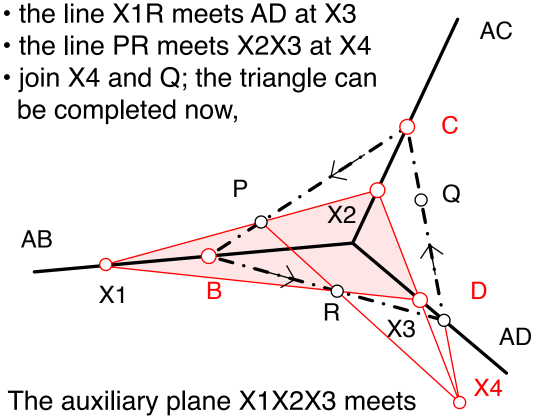


Fig. 2



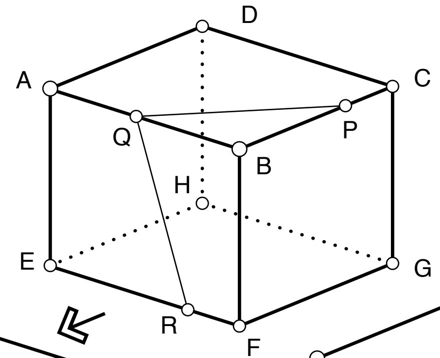
- choose X1 on AB
- the line X1P meets AC at X2
- the line X1R meets AD at X3
- the line PR meets X2X3 at X4
- join X4 and Q; the triangle can be completed now,



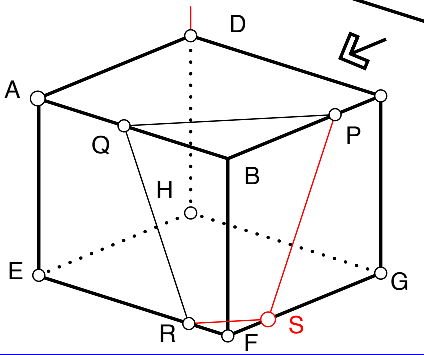
The auxiliary plane X1X2X3 meets the plane ACD at the line X2X3.The lines PR and X2X3 meet at X4.

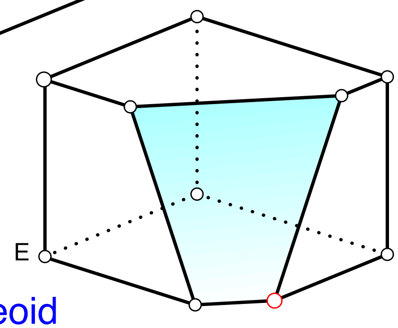
5.1 Truncate the "cub" (parallelepiped) ABCDEFGH with the plane POR.

• RS is parallel to PQ
• join P and S



Verification:
QR and PS meet
on BF

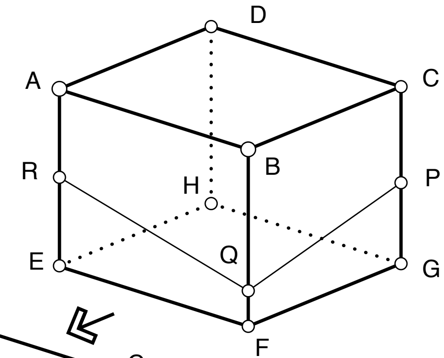




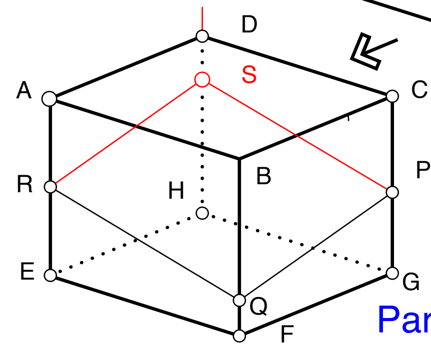
Trapezeoid

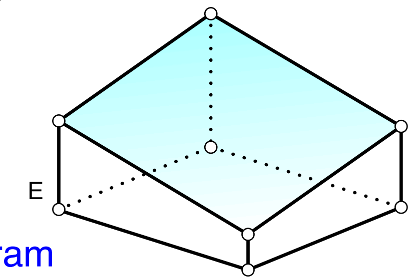
5.2 Truncate the "cub" (parallelepiped) ABCDEFGH with the plane POR.

• RS is parallel to PQ
• PS is parallel to QR



Verification:
PS and PQ meet
on CG

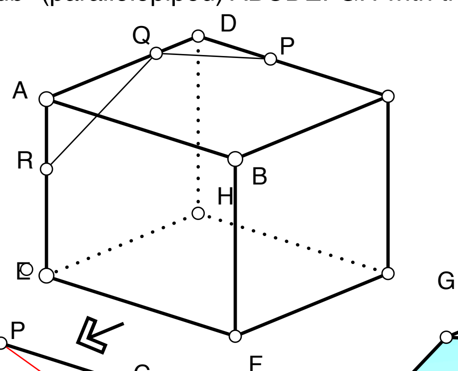




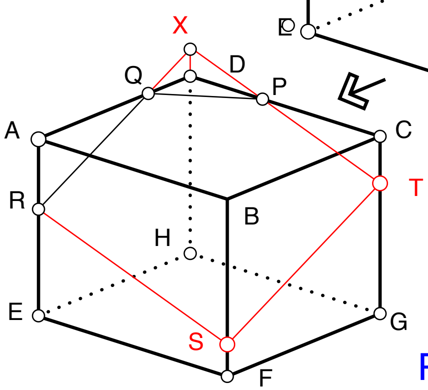
Parallelogram

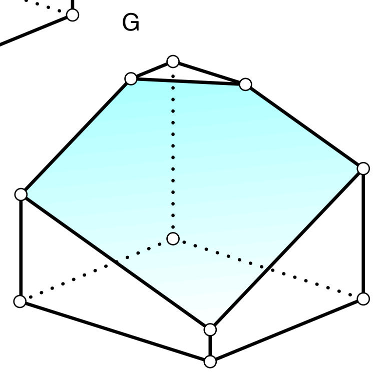
5.3 Truncate the "cub" (parallelepiped) ABCDEFGH with the plane POR.

• QR meets DH at X
• PX meets CG at T
• TS is parallel to QR



Verification:
RS is parallel to PT

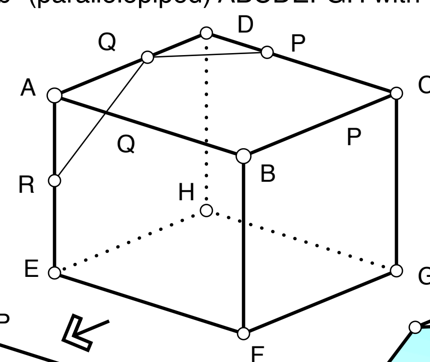




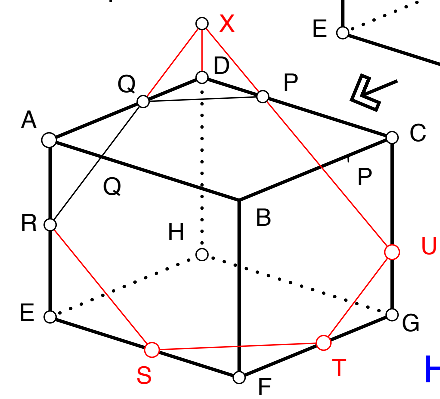
Pentagon

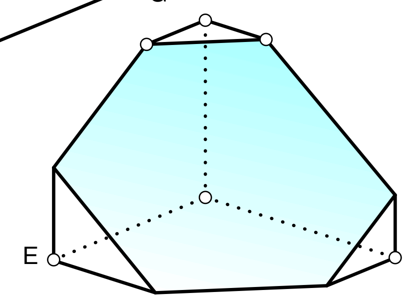
5.4 Truncate the "cub" (parallelepiped) ABCDEFGH with the plane POR.

• QR meets DH at X
• PX meets CG at U
• TU is parallel to QR
• ST is parallel to PQ



Verifications:
RS is parallel to PU
PU and ST meet
on GH

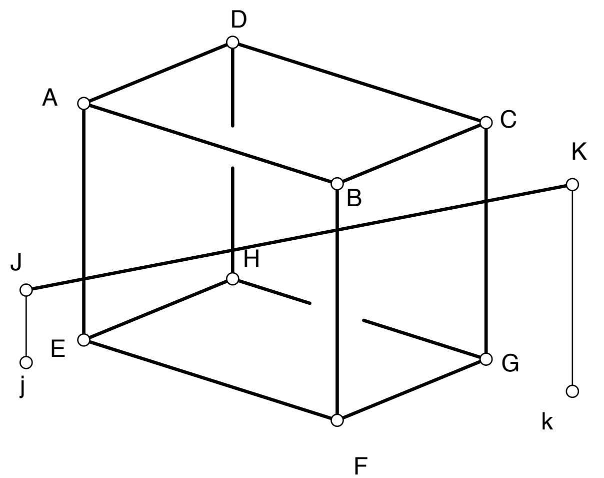




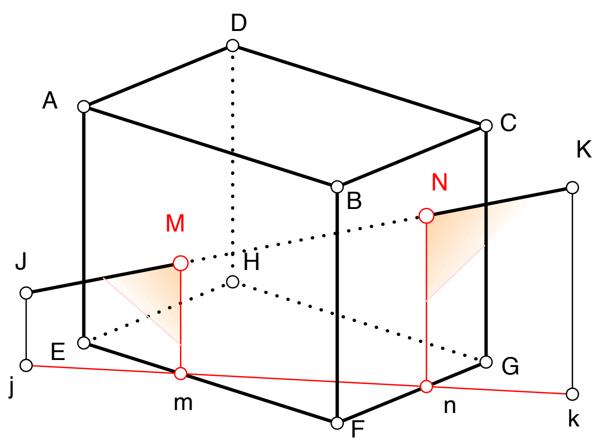
Hexagon

6. Problems I.

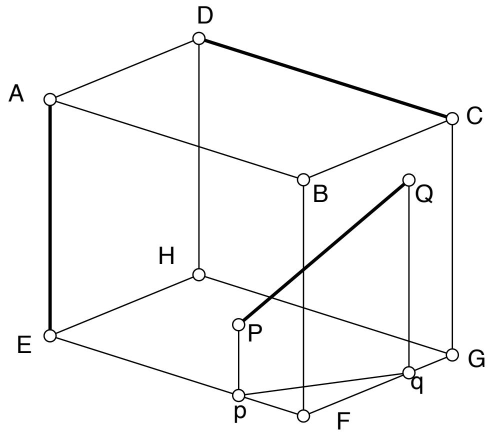
6.1 Given the cube ABCDEFGH and the line JK with its projection on the plane EFG. Construct the piercing points M and N of the line and the cube.



The auxiliary plane JKk intersects the plane EFG in the line jk.The line jk meets EF in m, fFG in n. Project m and n onto the line JK.



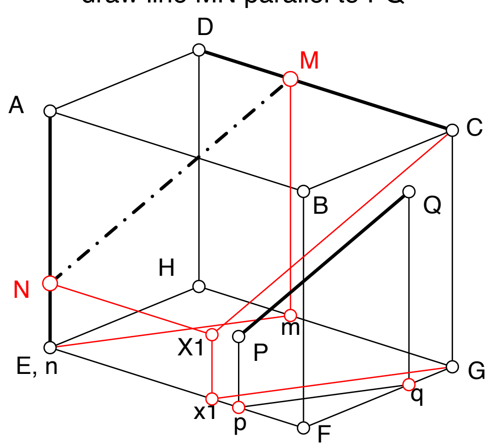
6.2 Given two skew lines, AE, CD and the direction PQ. P is in the plan ABF, Q is in the plane BCG Construct the line MN concurrent with the two skew lines and parallel to the direction PQ.



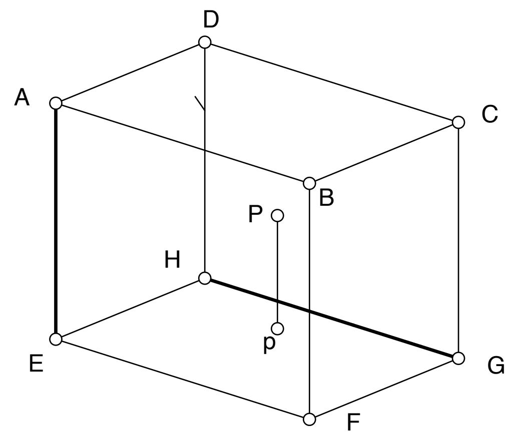
Point M is the piercing point of the line CD, and the pane defined by AE and the direction PQ.

Solution 1: • passe the line mn by E, parallel with pq
• the line mn meets GH in m
• project m into M on CD
• draw a parallel line to PQ through M.

Solution 2: • draw the parallel line Cx1 to PQ by C
• draw the projection Gx1 parallel to pq
• project x1 into the point N
• draw line MN parallel to PQ

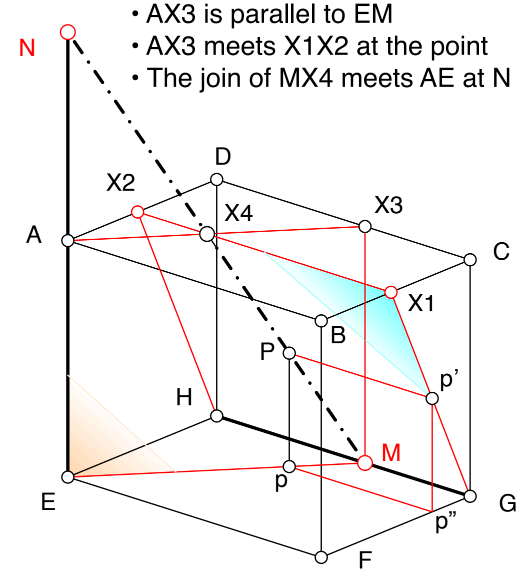


6.3 Given two skew lines AE, GH and the point P. Construct the line MN concurrent with the two skew lines and passes through the point P.



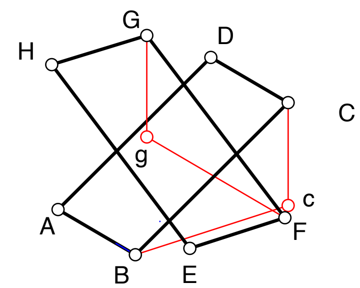
The line MN is the intersection of two planes: AEP and GHP.

Solution: • project P and p into p' and p"
• the join of G and p' meets BC in X1
• X1X2 is parallel to DC
• the join of Ep meets GH at M
• AX3 is parallel to EM
• AX3 meets X1X2 at the point
• The join of MX4 meets AE at N

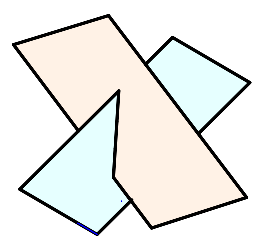


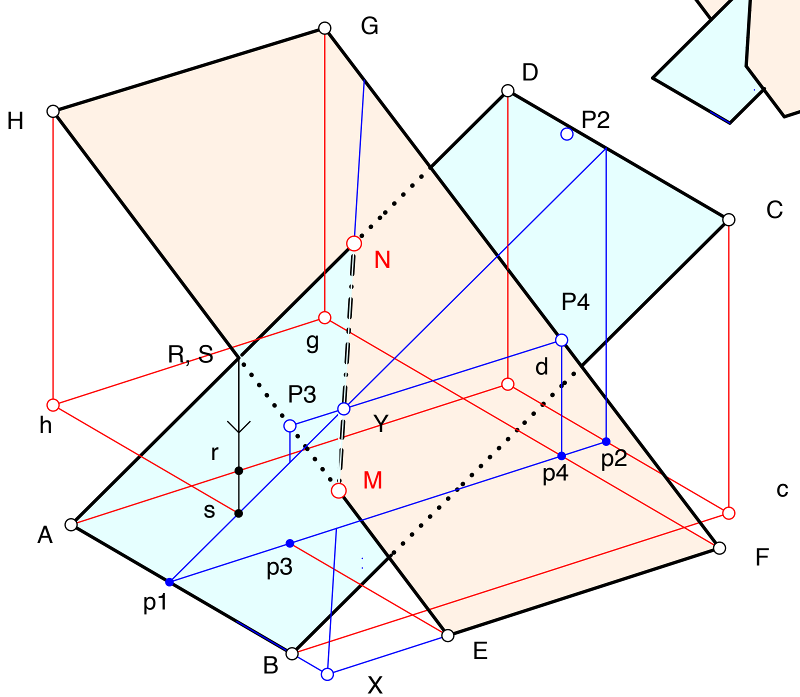
7.1 Given two parallelograms, ABCD and EFGH with their projections ABc and EFg. Construct their intersection MN.

The lines AB and EF are coplanar; their common point X is a point of the intersection. To find another point on MN, we pass an auxiliary vertical plane p1p2P2.This plane intersects with the plane ABCD in the line p1P2, with the planeEFGH in the line p1P2. These two lines meet at the point Y. Join X and Y; the intersection of the two planes is the segment MN, which lies inside the two contours of the two plane figures,



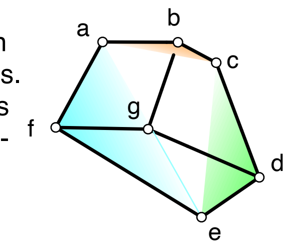
In our system, the visibility (the hidden lines) are determined. The right option (there are only two) is selected the following way: take the virtual intersection of two edges, AD and EH. The vertical line drawn from R,S will intersect the projection of AD first; therefore, AD is positioned above EH at this location.



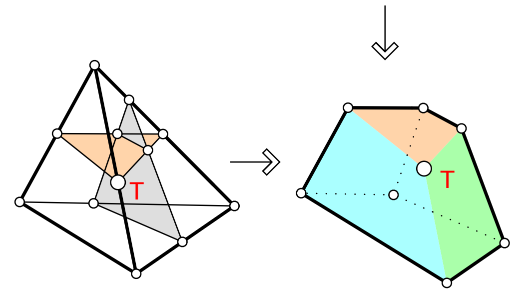


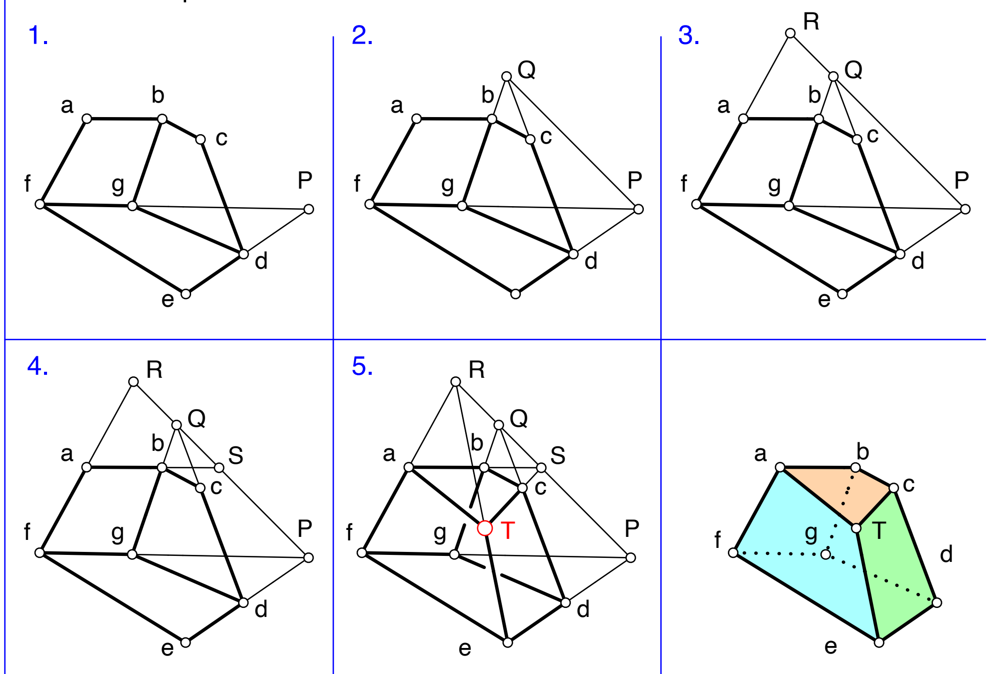
7.2 Given three planes abf, bcd and def. Construct the common point T of the other three planes abc, cde and efa. The three quadrilaterals abgf, bcdg and defg meet at a common point g.

Every convex (simply connected) polyhedron can be derived from a tetrahedron by successive truncations. As we will show below, here, a projection of the figure is sufficient (without a plan view) to determine the configuration. In the following, we will reconstruct the original tetrahedron efPR in five steps, which can be truncated into this new polyhedron (Fig. 6), called a hexahedron or an irregular cube.

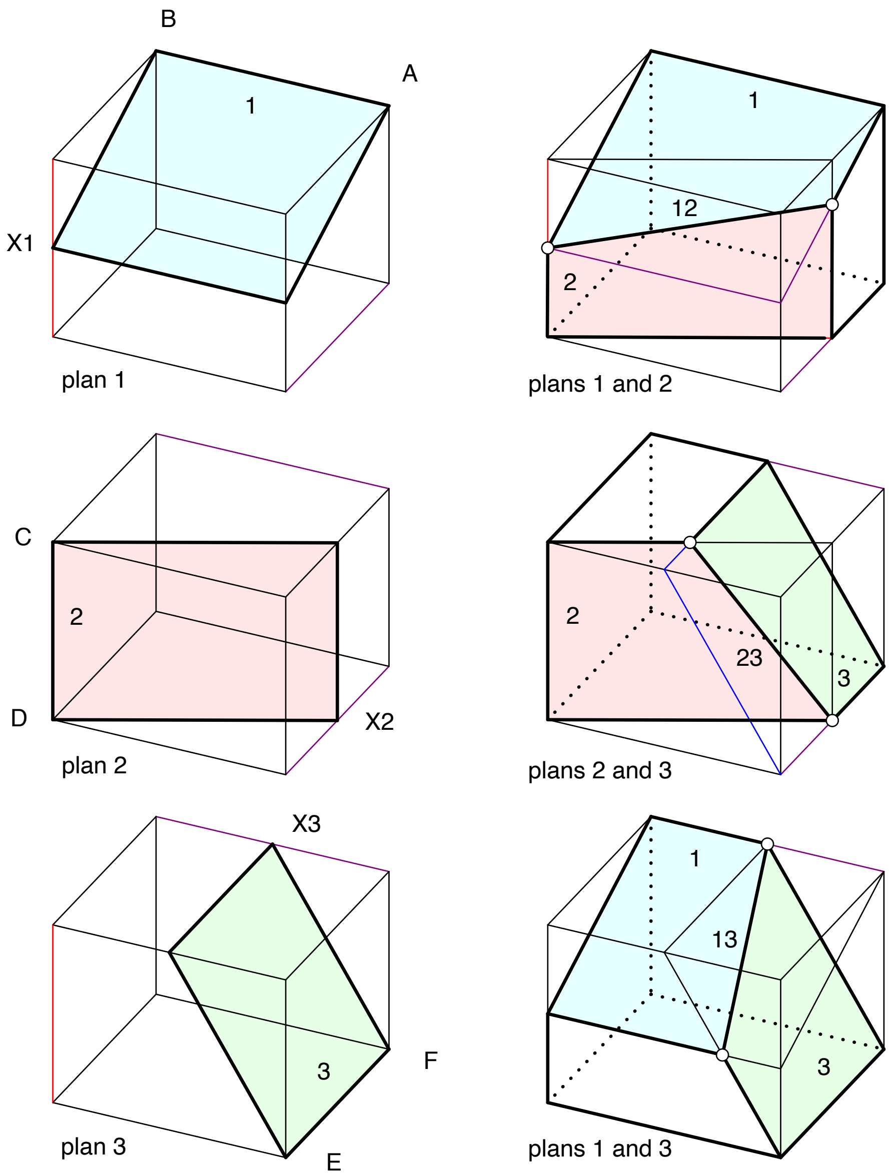


1. Extended fg and de meet at P
2. Extended bg and cd meet at Q
3. The join PQ and af meet at R
4. Extended ab meets PR at S
5. Extended cS meets eR at T, which is the common point of the three planes





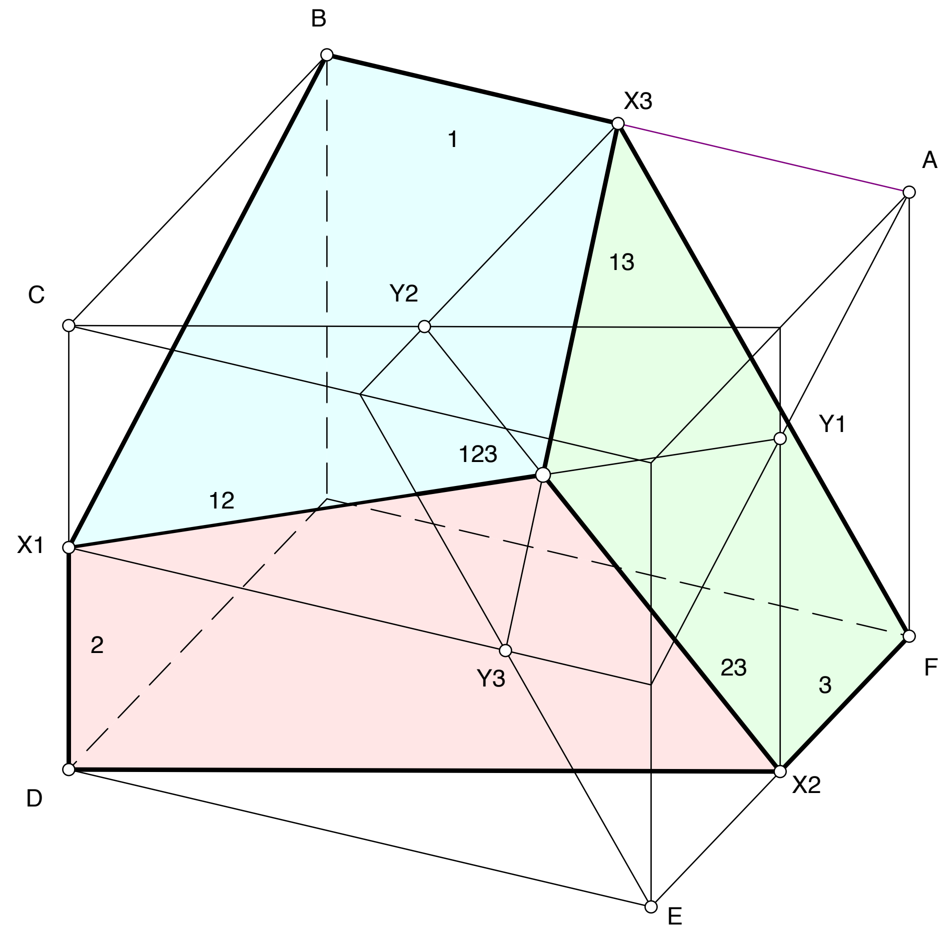
8.1 Given a “cube” (parallelepiped) and three planes, 1 and 2 and 3. Construct the solid (polyhedron - hexahedron) truncated by the three planes. We show below the three planes and their pairwise intersections 12, 13 and 23.



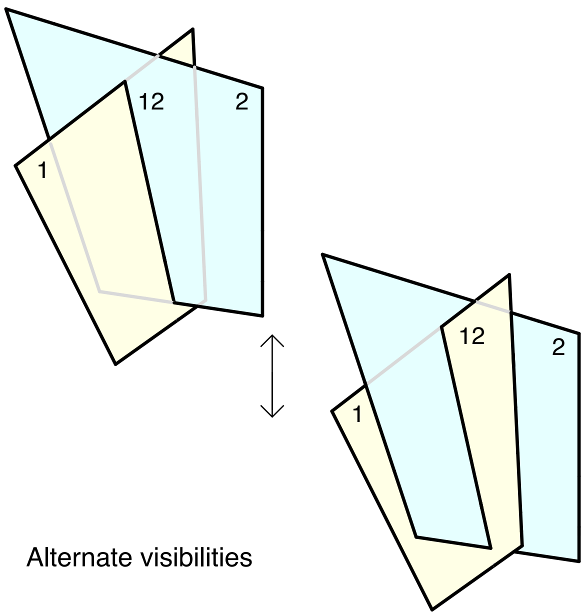
Three planes in a general position meet at the point 123, which is also the common point of the three intersections 12, 23 and 13.

- the plans 1 and 2 have two common points, X1 and Y1
- the intersection 12 is the join of the points X1 and Y1
- the plans 2 and 3 have two common points, X2 and Y2
- the intersection 23 is the join of the points X2 and Y2
- the plans 1 and 3 have two common points, X3 and Y3
- the intersection 13 is the join of the points X3 and Y3

The intersections of the three lines 12, 23 and 13 meet at the point 123. If you start with a regular cube (six identical square faces), and the points X1, X2 and X3 are midpoints of the respective edges; the three new faces will be identical quadrilaterals.

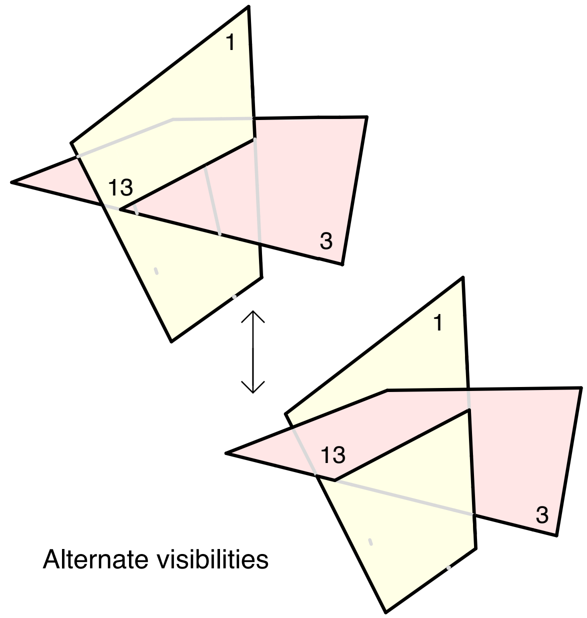


9.1 Intersection of two planes 1 and 2



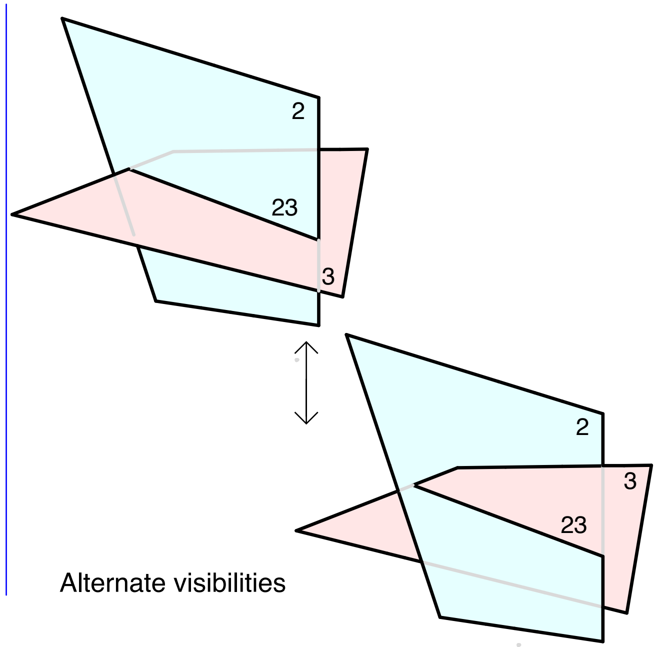
Alternate visibilities

9.2 Intersection of two planes 1 and 3



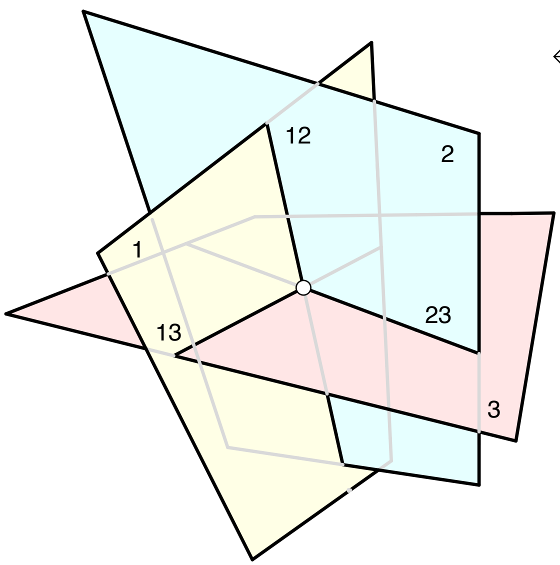
Alternate visibilities

9.3 Intersection of two planes 2 and 3

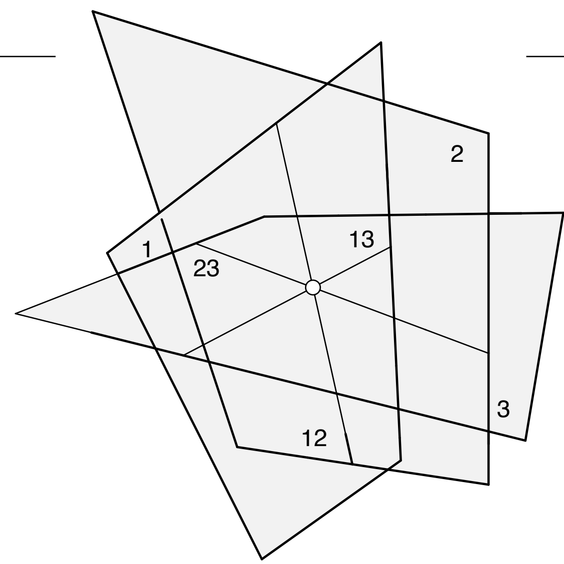


Alternate visibilities

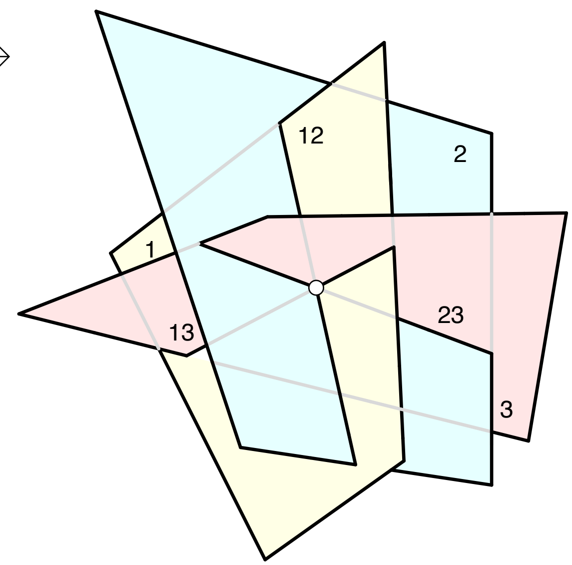
9.4 The common point of three intersecting planes 1, 2 and 3

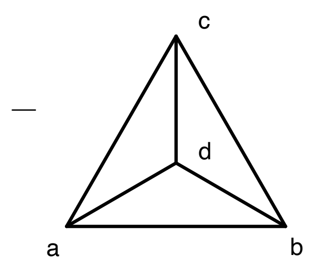


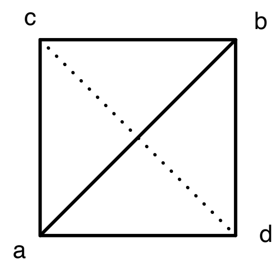
9.5 The three transparent planes 1, 2 and 3

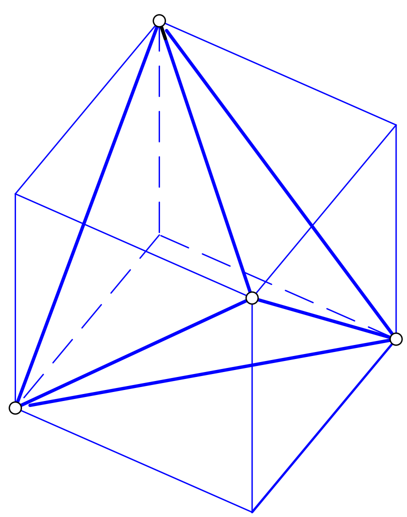


9.6 The common point of three intersecting planes 1, 2 and 3, alternate visibility







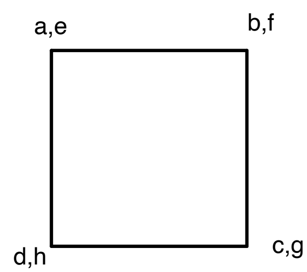


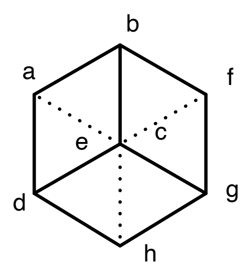
10.1 The six edges of the tetrahedron are the diagonals of the cube's faces

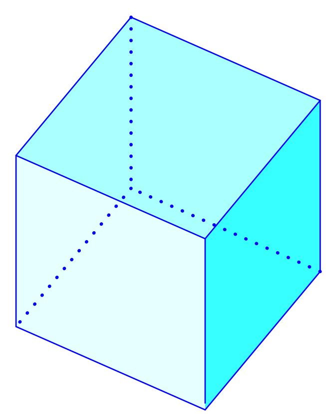
There are five regular polyhedra (Platonic solids), and they are strongly related by different means:

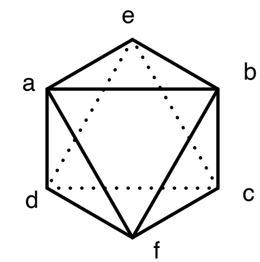
tetrahedron
octahedron
cube
icosahedron
dodecahedron

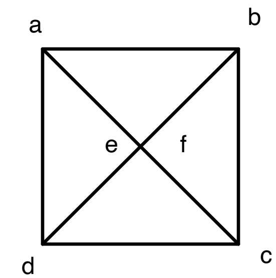
An interesting relationship: four polyhedra may be symmetrically inscribed in the cube, as shown below and on the following pages. Also shown are the two principal orthogonal projections of the three polyhedra.

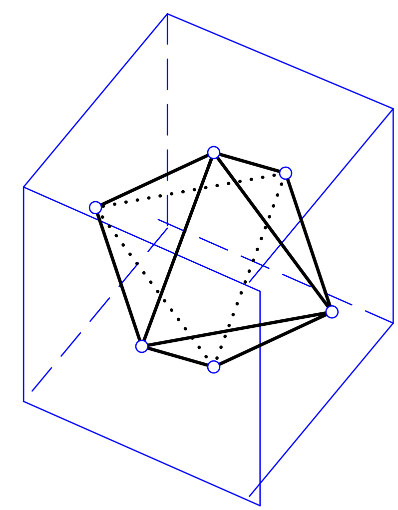








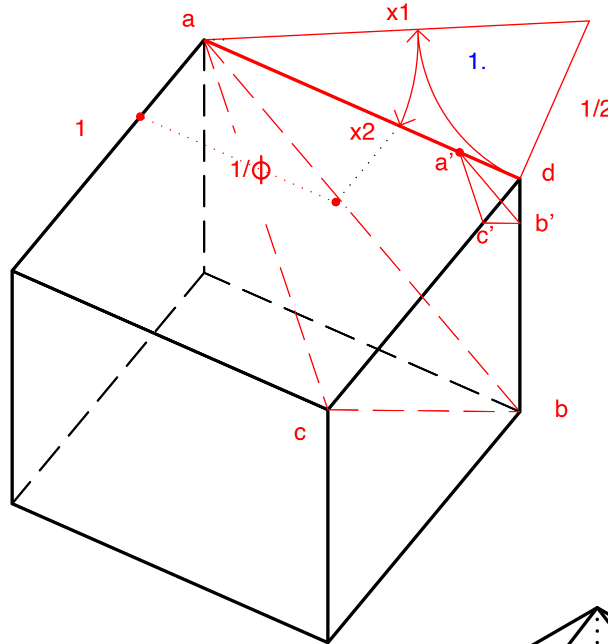


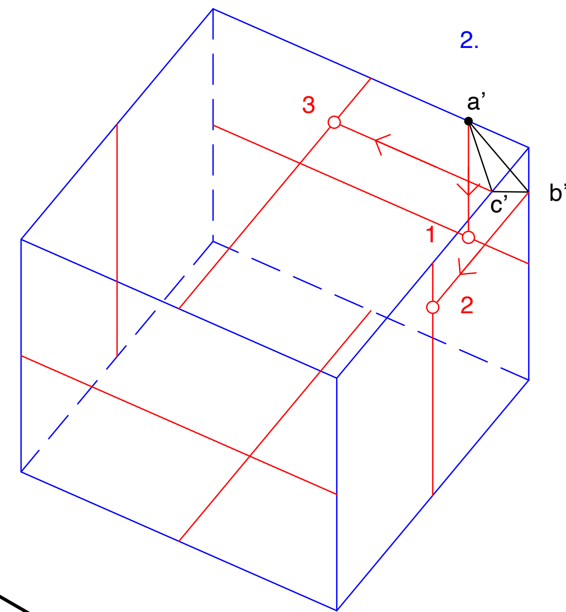


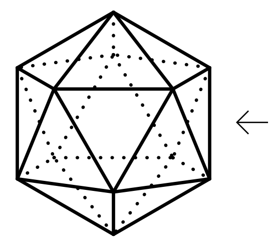
10.2 Six vertices of the octahedron are the centre points of the cube's faces

11.1 The regular icosahedron has 20 triangular faces, 30 edges and 12 vertices. When it is inscribed in a cube, six edges are in the cube's six faces (Fig. 4). We construct these edges whose endpoints are the 12 vertices of the icosahedron. If the cube's edge is 1, the edge of the icosahedron is 1/**Φ**, where **Φ**=1.618.

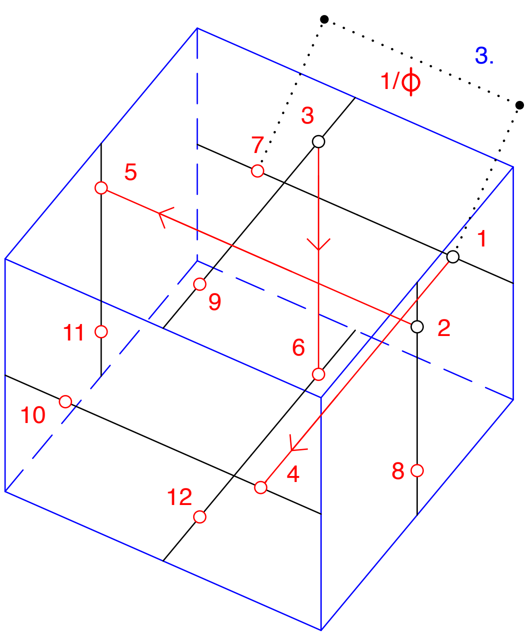
1. construction of 1/**Φ**, location of a', b', c'
a' b' is parallel to ab, etc, a is the midpoint of the distance x2d
2. projections of a', b', c' into vertices 1, 2, 3
3. 1, 2, 3 are projected into 4, 5, 6, the other 6 vertices are located on the corresponding edges symmetrically
4. the six edges (and the twelve vertices) are shown
5. the remaining 24 edges are inserted

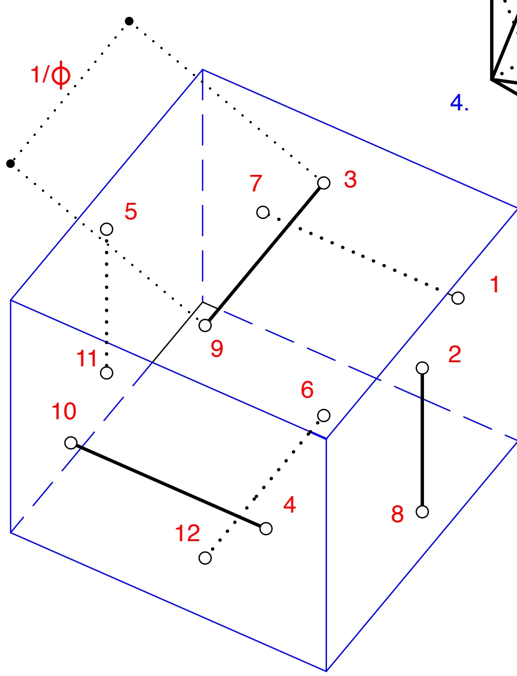


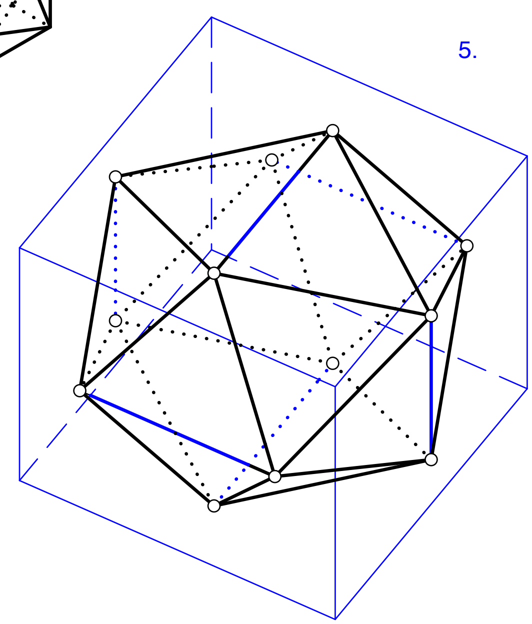




orthogonal projection displays the 3-fold axis



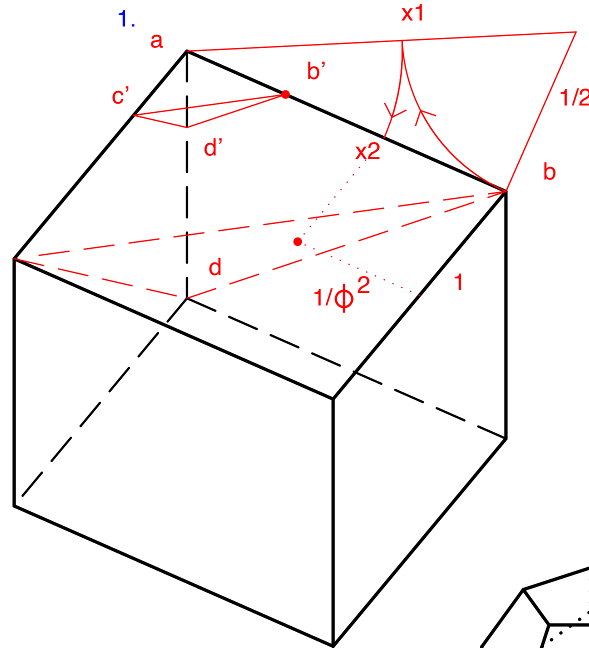




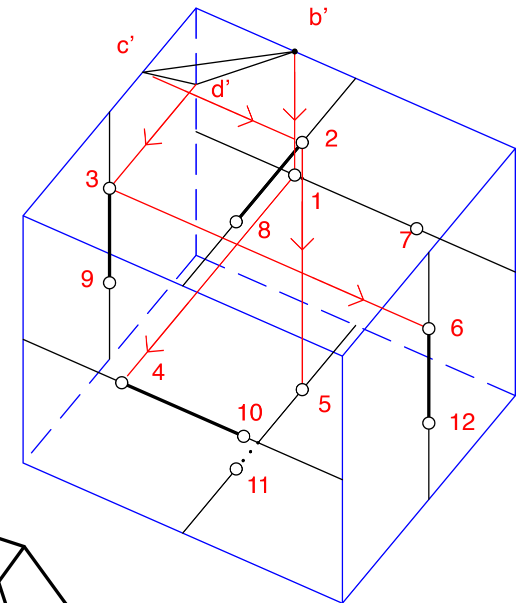
12.1 The regular dodecahedron has 12 pentagonal faces 30 edges and 20 vertices. When it is inscribed in a cube, six edges are in the cube's six faces (Fig. 2). We construct these edges whose endpoints are 12 vertices of the dodecahedron. If the cube's edge is 1, the edge of the icosahedron is $1/\Phi^2$, where Φ=1.618.

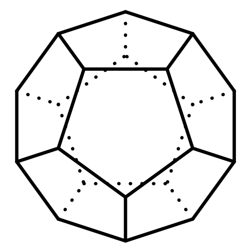
1. construction of $1/\Phi^2$, location of b', c', d', b'c' is parallel to bc, etc, b' is the midpoint of the distance ax2
2. projections of b', c', d' into vertices 1, 2, 3; they are projected into 4, 5, 6, and the other six vertices are located on the corresponding edges symmetrically; six edges (and twelve vertices) are shown here
3. knowing the vertices 4, 8, and 10, the vertices 13 and 14 are constructed using the Golden Ratio, all others are determined by parallelism between edges and diagonals
4. rotating the visible edges within the contour 180° about the center C of the cube will reveal the hidden edges
5. the complete dodecahedron

1.



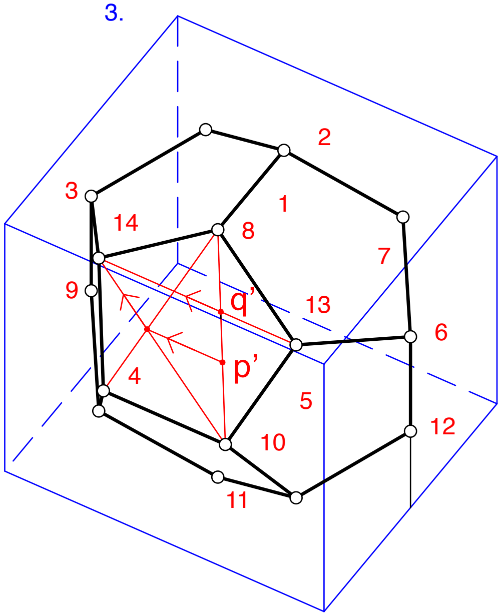
2.

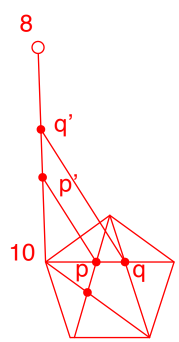




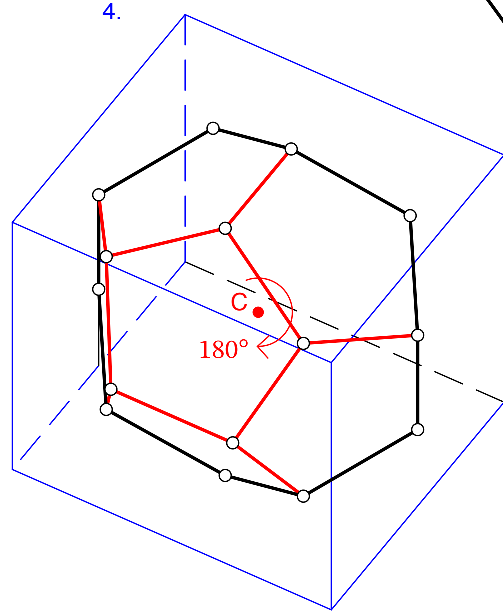
← orthogonal projection displays the 3-fold axis

3.

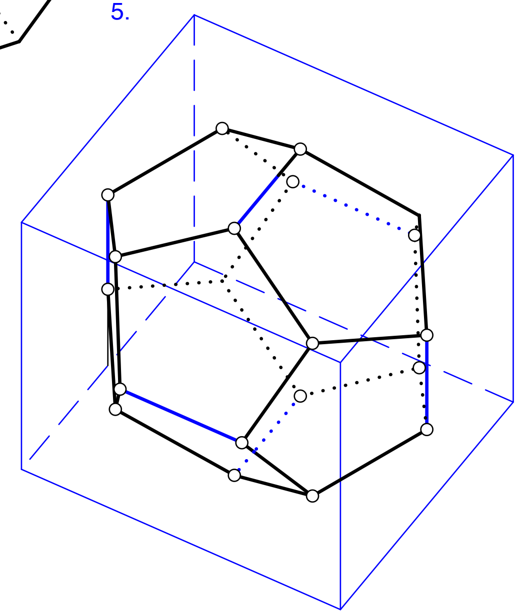




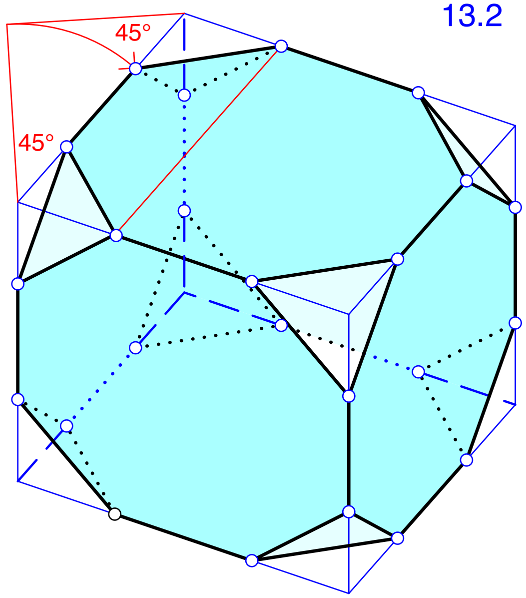
4.



5.

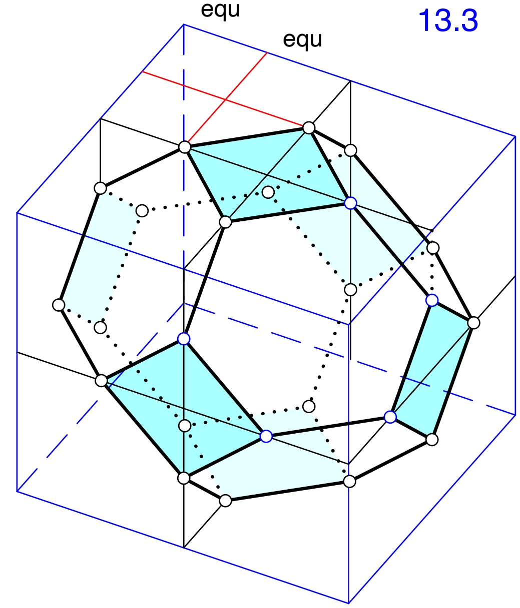


13.2

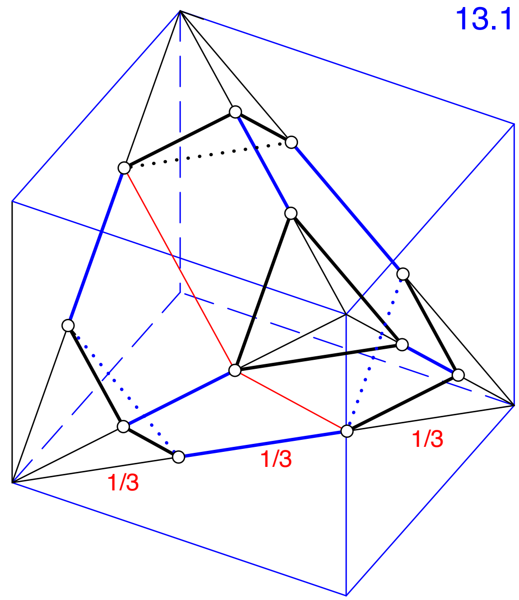


On the following pages, we present seven of the thirteen semiregular polyhedra. They exhibit four distinct symmetry groups. The tetrahedron has 2 and 3-fold rotational axes; five have 2, 3, and 4-fold axes, and the other five have 2, 3 and 5-fold axes. The two others (the snub cube and the snub dodecahedron) have 2, 3, 4 and 2, 3, and 5 fold axes but no mirror symmetries.

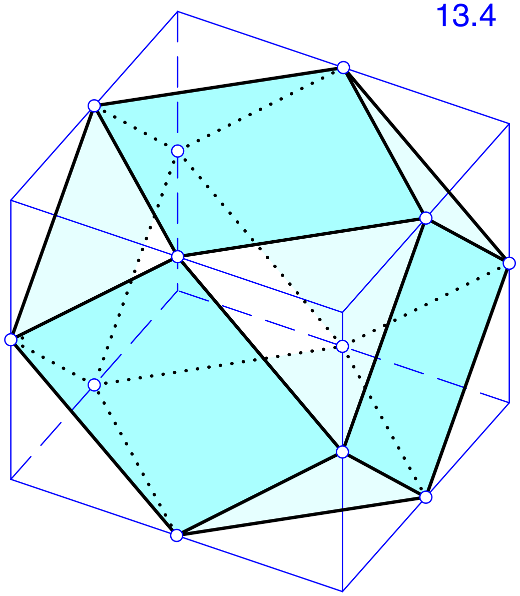
13.3



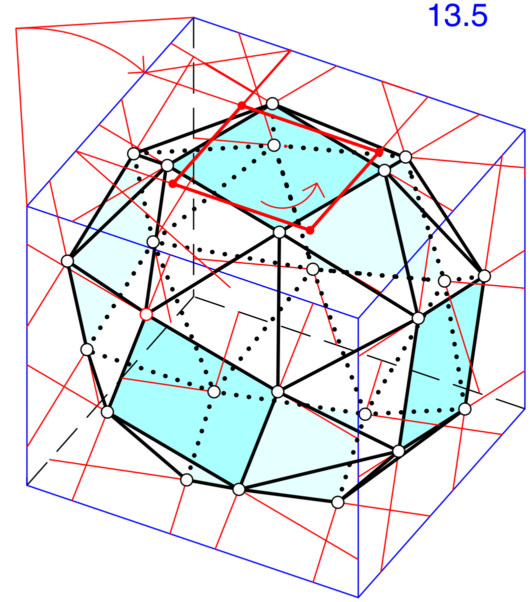
13.1



13.4



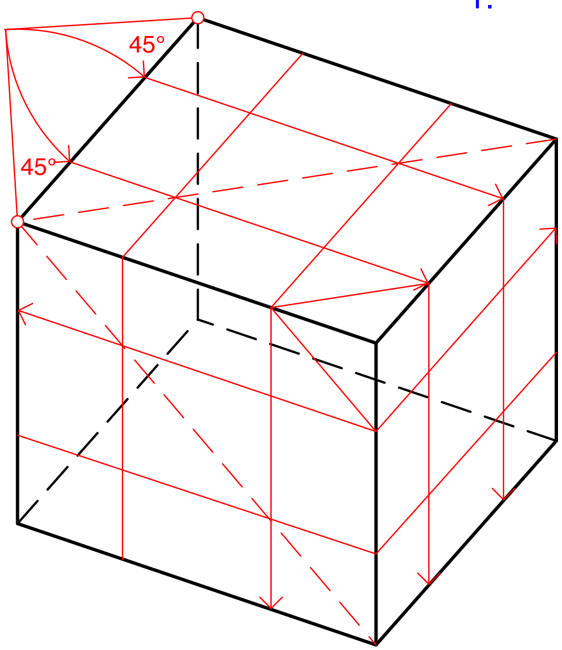
13.5



13.1 Truncated tetrahedron: 12 vertices, 18 edges, 8 faces; six edges are on the diagonals of the cube's six faces.
13.2 Truncated cube: 24 vertices, 36 edges, 14 faces.
13.3 Truncated octahedron: 24 vertices, 36 edges, 14 faces.
13.4 Cuboctahedron: 12 vertices, 24 edges, 14 faces.
13.5 Snub cube:24 vertices, 60 edges, 38 faces. The square faces are rotated 16.468°

14. Rhomicuboctahedron

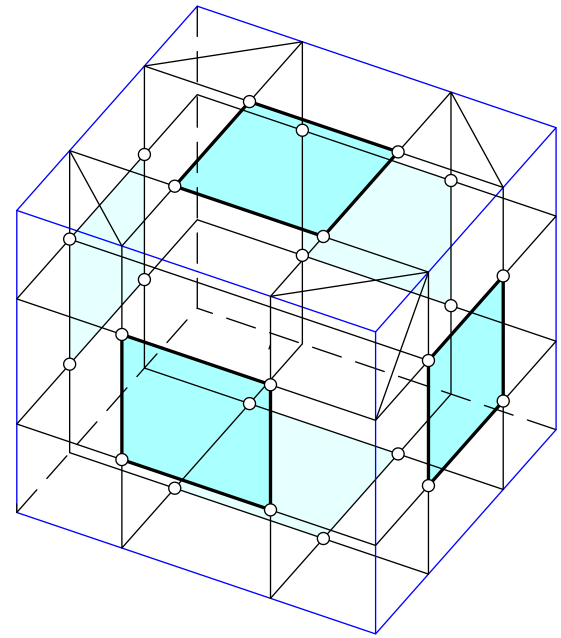
1.



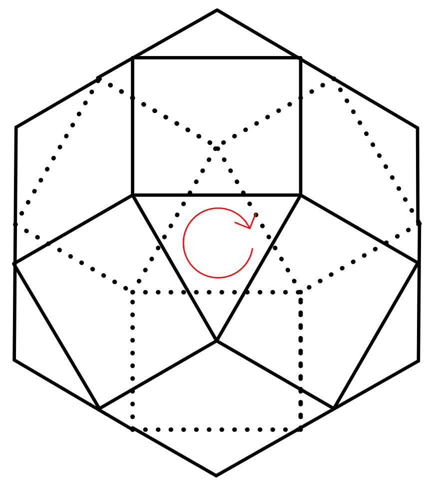
The rhombicuboctahedron has 24 vertices, 48 edges and 26 faces, six inscribed in the six faces of the cube.

1. the parallel diagonals of the octagon are located with the help of a 45° triangle in the three visible faces of the cube
2. the squares are drawn in the six faces of the cube

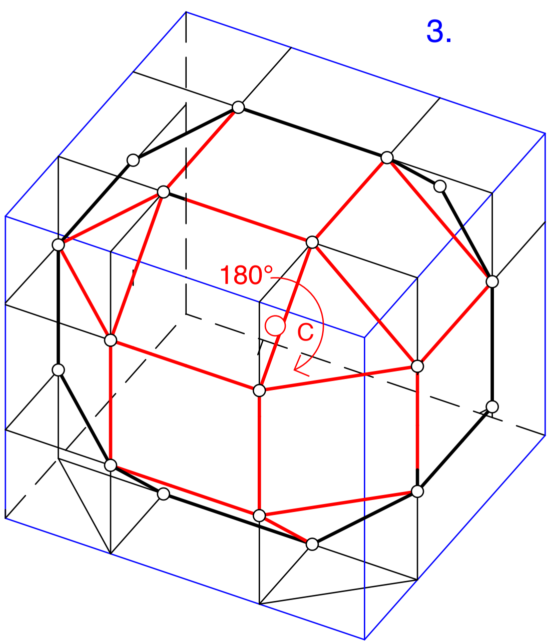
2.



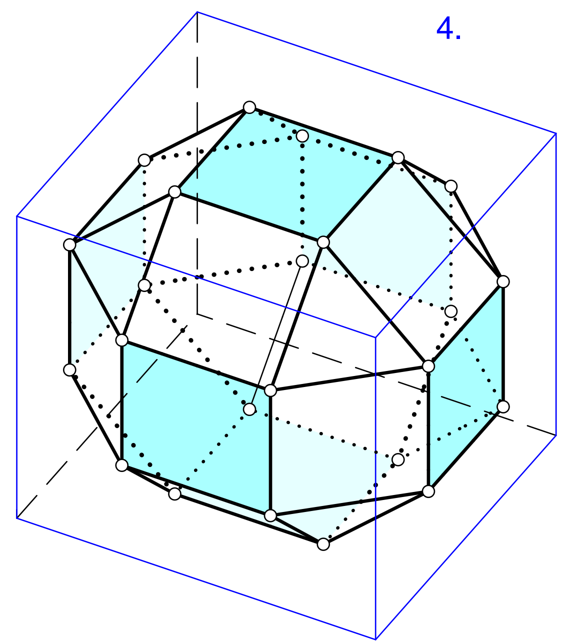
5.



3.



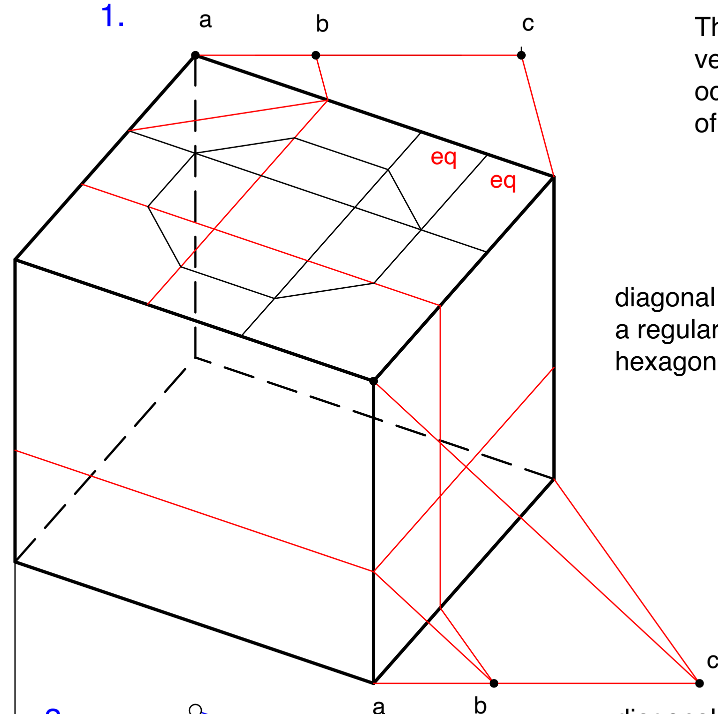
4.



3. the contour edges and the visible edges (red) are drawn
4. rotating the visible edges within the contour 180° about the center C of the cube will reveal the hidden edges
5. the orthogonal projection of the rhombicuboctahedron displays the 3-fold rotational axis.

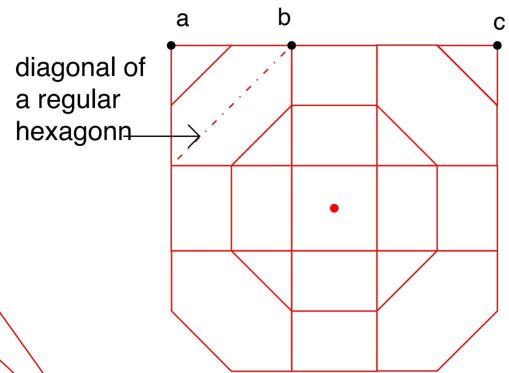
15. Truncated cuboctahedron

1.

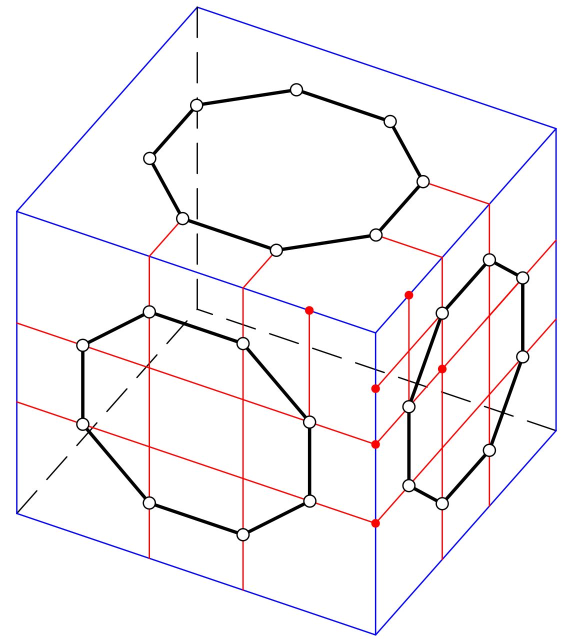


The truncated cuboctahedron has 48 vertices, 72 edges and 26 faces, six octagons are inscribed in the six faces of the cube.

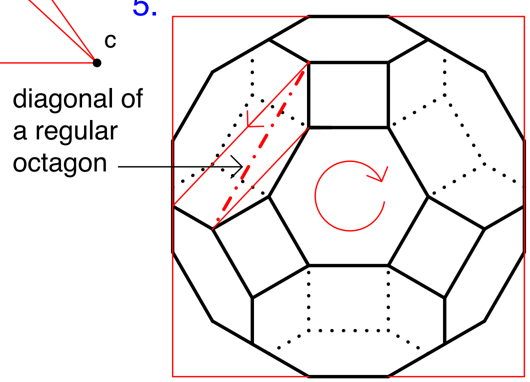
1. the parallel diagonals of the ctagon are located with the help of ratios



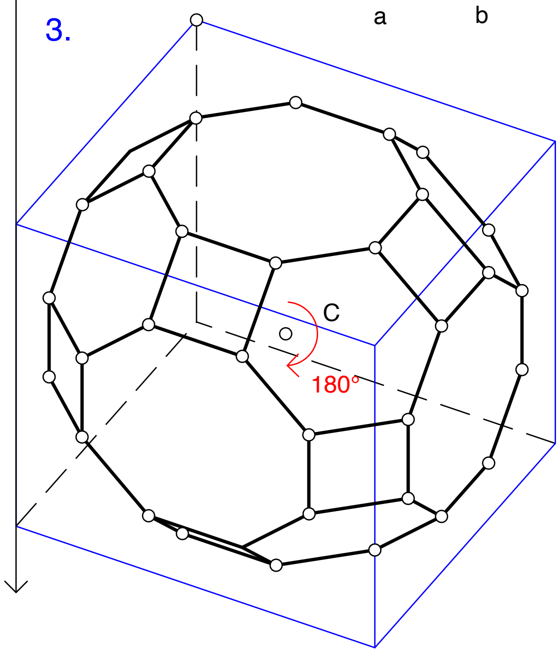
2.



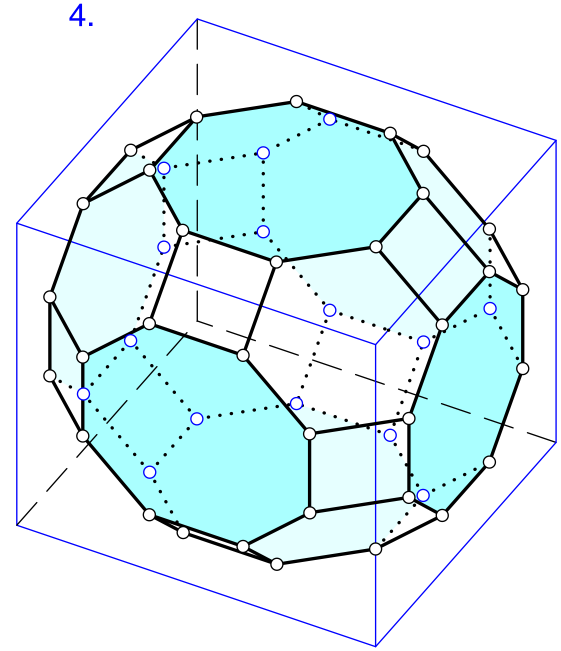
5.



3.

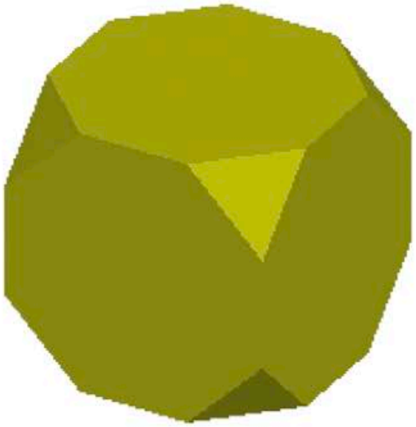


4.

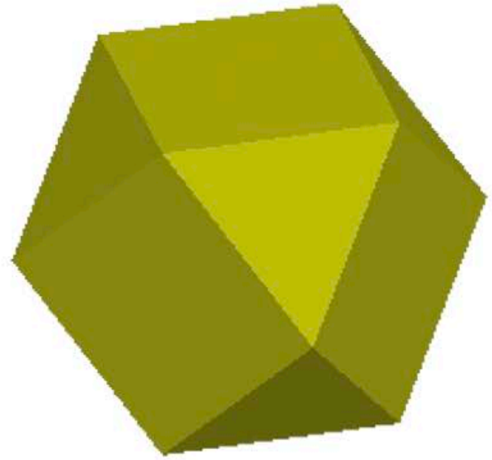


2. the octagon is installed in the visible faces
3. the octagons are connected, and the conour is completed.
4. rotating the visible edges within the contour 180° about the center C of the cube will reveal the hidden edges
5. the orthogonal projection of the truncated cuboctahedron, displaying the 3-fold rotational axis.

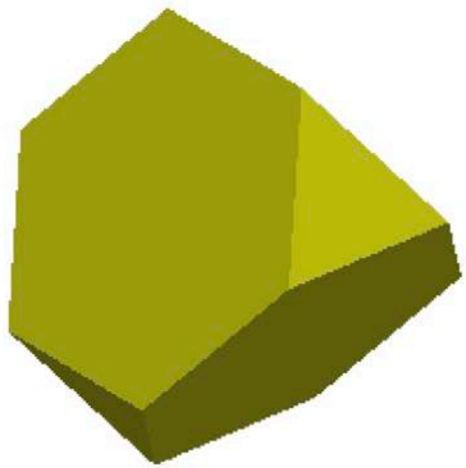
Cube tronqué



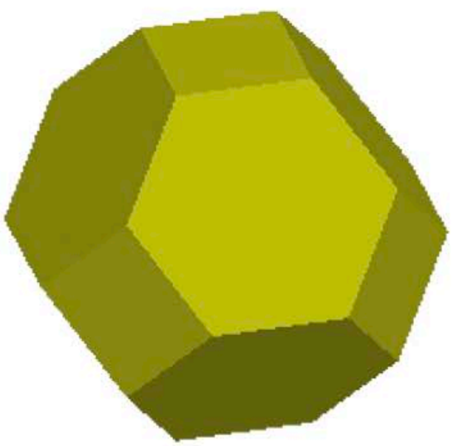
Cuboctaèdre



Tétraèdre tronqué



Octaèdre tronqué



Rhombicuboctaèdre