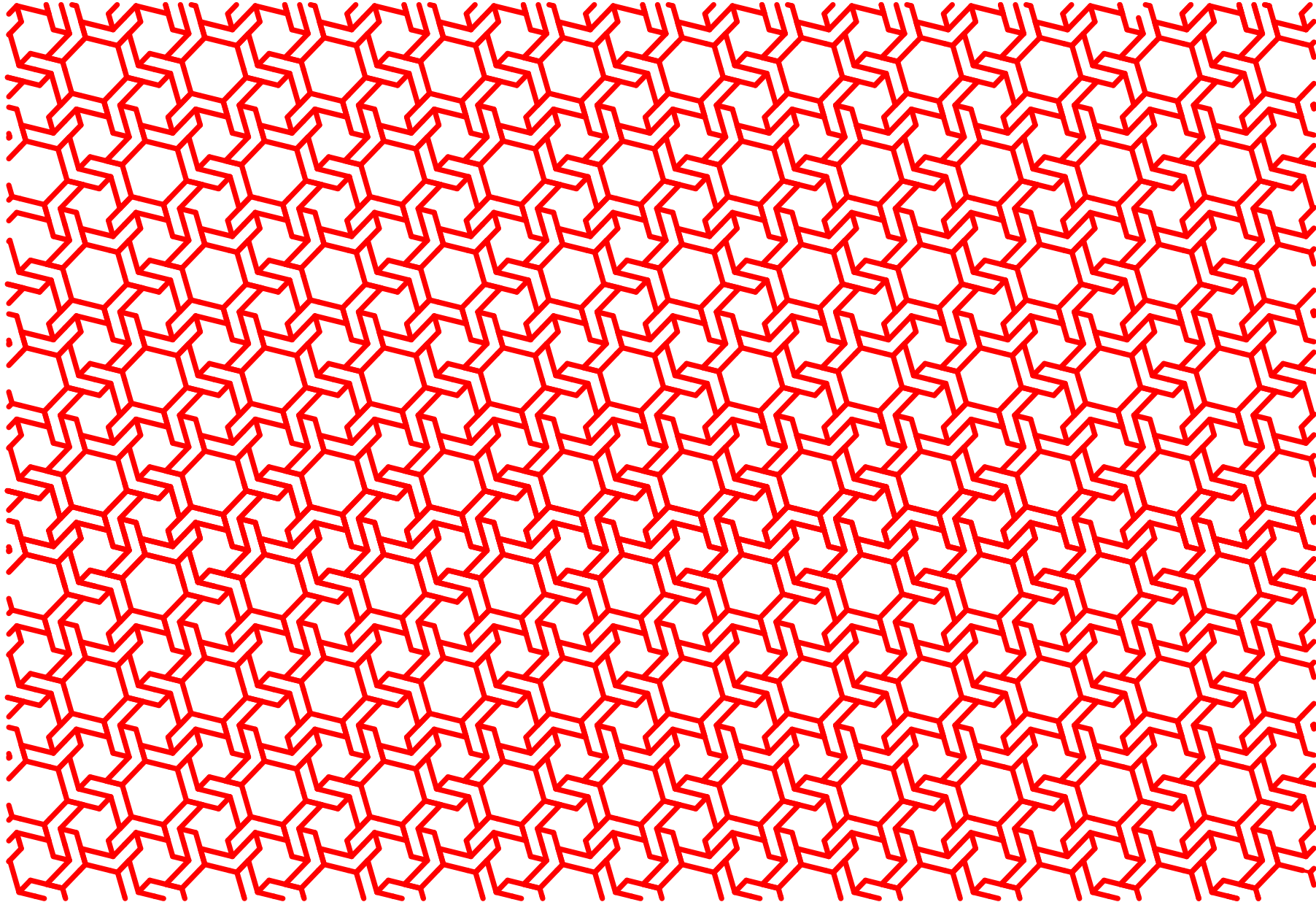




geometria2

# Symmetry 101

Plato:  
Beauty is  
bound up  
with symmetry



Symmetry, an abstract chapter of modern mathematics, is strongly linked to the organic and inorganic nature and to visual art. The digital revolution plays a vital role today: it opened the door to the pleasure of discovering a great variety of patterns never seen before as well to analyze artworks created in history.

Our educational system ignores the subject of symmetry. The common knowledge is mainly limited to the bilateral (mirror) symmetry; the left and right hands. It does not help that most books on this subject are written by mathematician for mathematicians.

We choose the title for this volume, “Tiling with Geometria.” Geometria is an all-purpose drafting and graphic software we are presently completing. Every page in this document was prepared with this software in its beta version. The main objective of Geometria is a very fast learning curve and intuitive application of symmetry operations.

This short volume is meant to complement Geometria. The basics of symmetry are followed by the presentation of symmetry groups and completed by different methods of tilings. We skip the mathematics and rely on the visual approach while maintaining a certain rigour with the terms and definitions. However, Tiling with Geometria is not meant to be a user guide to the software. The following pages do not contain any reference to the tools of Geometria, we refer the reader to the Help function in the software. There is a price to pay for the simplicity of Geometria. It requires a basic knowledge of middle-school geometry, which we believe is also necessary for intelligent use of computer graphics.

With this volume and Geometria, we aim to attract young and old to explore the symmetry domain. It is a wonderful experience of coordination between the brain, the eye and the hand, whether it controls the pencil or the mouse. We tested Geometria with young children and observed a keen interest and a desire to create patterns.

Natural growth in the inorganic world (crystallography) and in the organic nature (biology, botany, physiology, etc) is controlled by symmetry; a vast bibliography is available on this subject.

Two terms are the fundamental theme of symmetry: *transformation and invariance*. This seems like a paradox: changing a subject while the subject remains unchanged. We hope going through these pages will clarify this concept.

Sir Owen Jones, the British architect, published his book *The Grammar of Ornament* in 1856. In it, he made this remarkable statement:

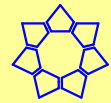
*“However **varied** the manifestation... the leading ideas on which they are based are **very few**”.*

#### Acknowledgement

The preparation of this document could not have been made possible without the software Geometria, the result of our cooperation with my devoted and competent friend **Louis Langlois**



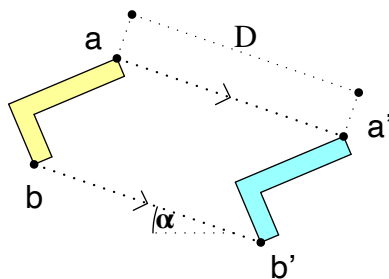
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| 20. Mirror tiles                       | 40 | of the content always precedes the Plate.                  |        |



# 1. Basics of symmetry in the plane

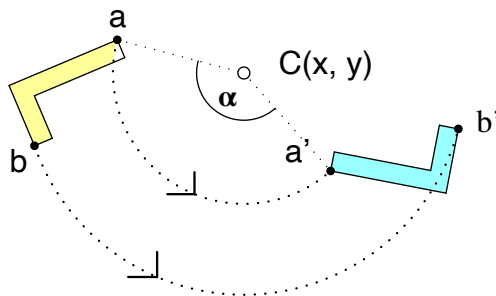
The four symmetry operations are:

## 1. translation



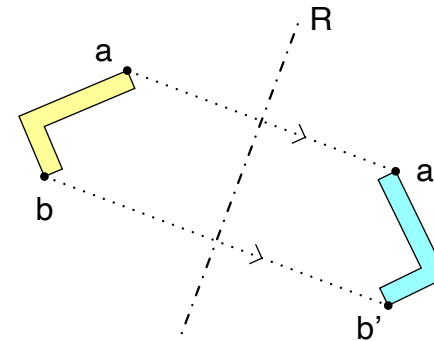
Translation is a parallel displacement of every point of a motif (the letter L), defined by a vector (distance  $D$  and angle  $\alpha$ ).

## 2. rotation



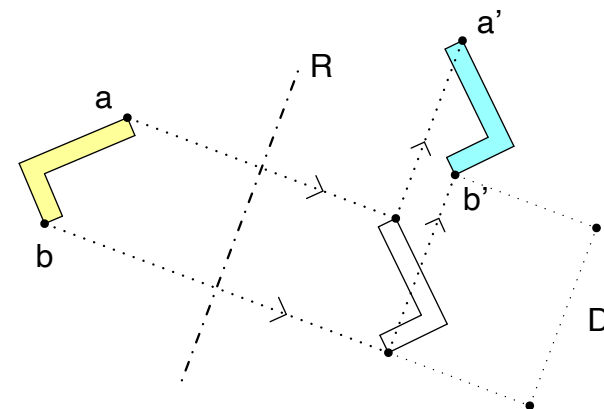
Rotation is a circular displacement of every point of a motif (the letter L), defined by the centre  $C$  and the angle  $\alpha$ .

## 3. reflection

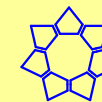


Reflection creates the mirror image of every point of a motif, defined by the situation of the axis (mirror)  $R$ .

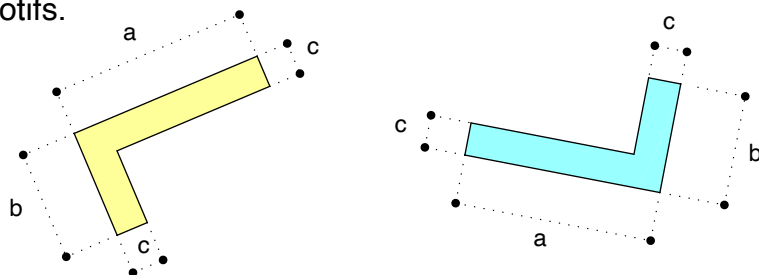
## 4. glide reflection



Glide reflection creates a temporary mirror image of a motif, which is then translated by a distance  $D$ , parallel to the axis  $R$ .



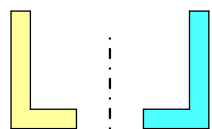
Transformations and invariance are the two key worlds in symmetry operations. On the previous page, the yellow motif is **transformed** into the blue motif, while the geometric properties (parameters) remained **invariant** for the two motifs.



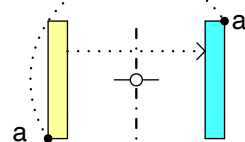
Translation and rotation are called direct motions (symmetries), reflection, and glide reflection are opposite motions. In opposite motion, the original and the displaced motif can not be reunited by congruence unless you “flip” one motive in 3-dimension. Similarly, your left shoe only becomes your right shoe if you can flip it in 4-dimension (!).

Two motifs are said to be symmetrical (isometric) if all parameters of one is identical to the parameters of the other. It is interesting to note that in the common language, only reflection is recognized as a symmetry operation.

Why did we choose the letter L as our motif? The motif itself should be asymmetrical (and simple) otherwise some of operations would be indistinguishable:



reflection



reflection, translation or rotation?

The basic theorem of symmetries:

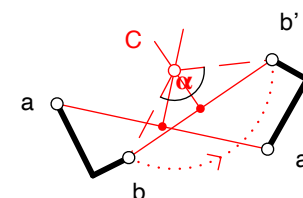
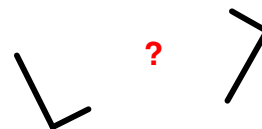
**Any displacement in the plane is either a rotation or a glide reflection.** Translation may be considered as a rotation of  $0^\circ$  angle around a center at infinity.

Here are two important constructions related to this theorem. In both cases, two congruent motifs are given. Before applying the construction, a visual inspection should confirm whether rotation (1) or reflection (2) is present.

### 1. Direct symmetry

Given two congruent motifs, find the center  $C$  and the angle  $\alpha$  of the rotation  $bb'$ ,

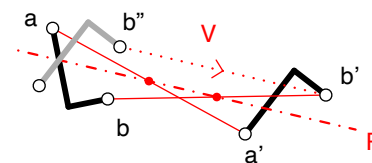
The center  $C$  is the intersection of the bisectors  $aa'$  and  $\alpha$  is the angle  $bCb$ .

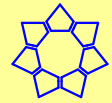


### 2. Opposite symmetry

Given two congruent motifs, find the axis of the reflection and the vector of the translation

The axis  $R$  is the line joining the midpoints of  $aa'$  and  $bb'$ , the vector  $V$  is the distance  $b'b''$





## 2. Symmetry groups in the plane PI #1

Up to now, we were preoccupied with the four symmetry operations and their effects on two congruent (isometric) motifs. We will now extend our interest to discuss how to organize and classify many or an infinite number of motifs into a **structurally coherent** system called groups. Group theory is an abstract chapter of higher mathematics with esoteric definitions and proofs so that we will limit our study of symmetry to the more visual and intuitive concepts.

Symmetry groups are classified by the number of distinct translations used to multiply the motif:

- point group - 0 direction (2 x  $\infty$ )
- line group - 1 direction (7)
- plane group - 2 directions (17)

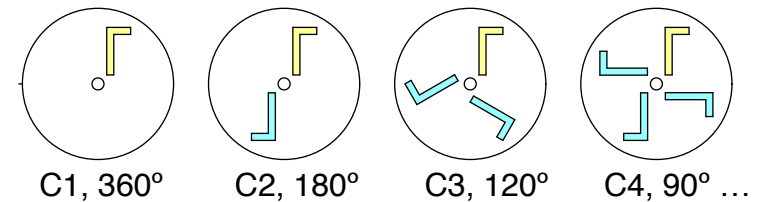
The numbers in brackets are the number of possible cases in each category with distinct structural rules. What is surprising, that for the linear and the planar groups, so few rules (7 and 17) can create such a wealth and diversity of patterns.

## 3. Point groups PI #1

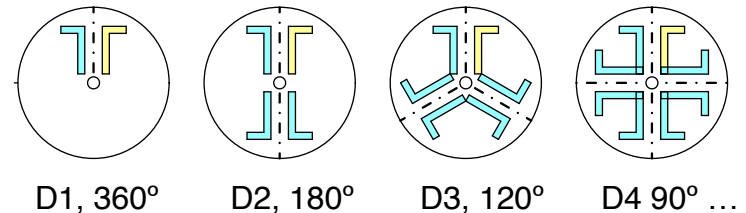
Point groups, also called rosaces from architectural practice, are divided into two subgroups: **cyclic** group ( $C_n$ ) with only rotation, **dihedral** group ( $D_n$ ) with rotation and reflection.

Examples:

Cyclic groups

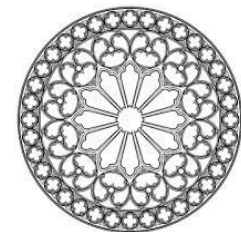


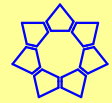
Dihedral groups



The angle of rotation for point groups is limited to  $360^\circ/n$ , where  $n$  is a positive integer. The rotated motive by  $360^\circ$  will coincide with itself, it is called identity. This is true for any rosace: if it is rotated by an angle of  $360^\circ/n$ , it will coincide with itself: transformation and invariance!

Rose window is often used as a generic term applied to a circular window, found in many Gothic cathedrals and churches in Northern France

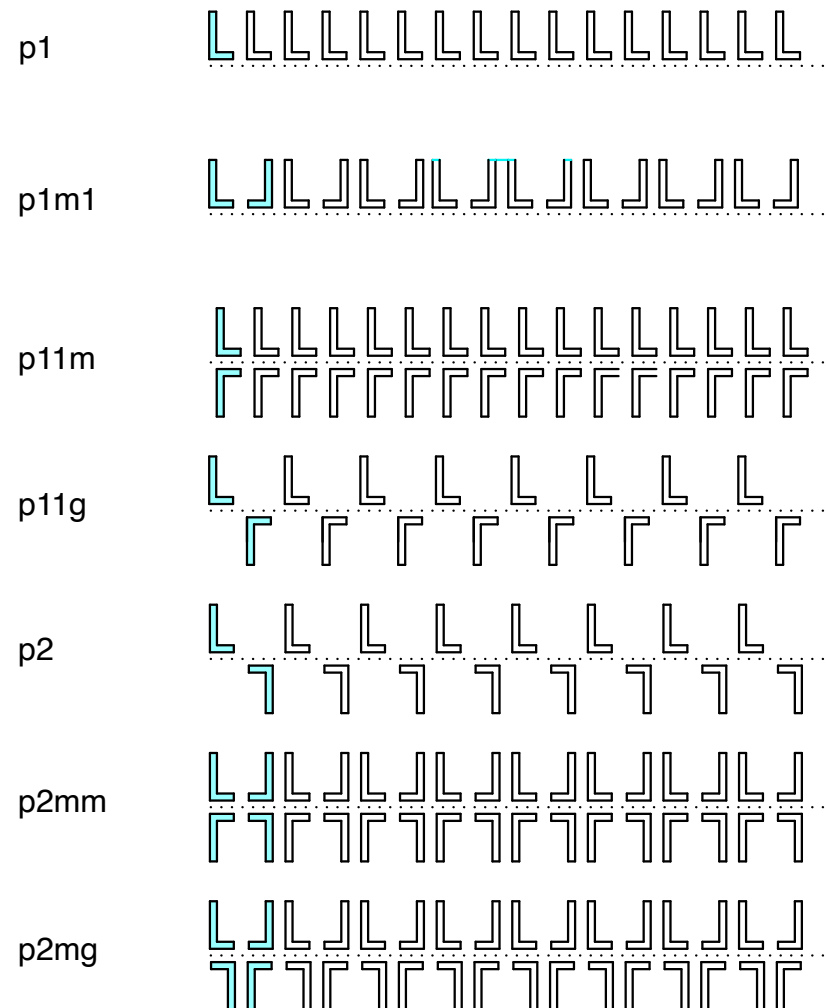




## 4. The 7 line groups

PI # 1

The seven line groups are depicted below:



Since translation is permitted only in one direction, the only the following other symmetry operations are permitted:

- 2-fold rotation ( $180^\circ$ )
- reflection parallel or perpendicular to the axis
- glide reflection parallel to the axis

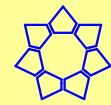
We used the letter L as a motif again. The simplest non symmetrical motif is essential to reveal clearly the structure of the operations. A more complex and maybe symmetrical motif would certainly result in a visually more pleasing solution.

We introduce here the concept of the fundamental region: the fewest motifs, which when translated, can produce the group. In the case of the simplest one, p1, a single motive is the fundamental region. Two motifs are required for the following four linear groups, while for the last cases, we need four motives. In order to draw groups or patterns quickly on a computer, the method of translating the fundamental region is the fastest and most efficient.

On [PI #1](#), we also show the unit cell and the generators together, they provide the minimal information to draw the group. The unit cell contains the motif and its position relative to the operators; the generators show only the operations.

Frieze is often used as a generic term for line groups due to their architectural applications.



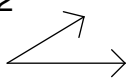
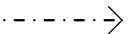
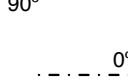
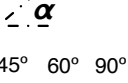
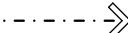
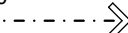
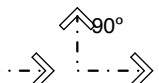


## 5. The 17 plane groups

PI 1

If we want to multiply a motif in the plane, arrange their positions and orientations in a periodical, “regular” manner by using one or more of the four symmetry operations. There exists 17 distinct combinations, proven by Fedorov, a Russian crystallographer (1891).

Plane groups also called “wallpaper groups” are defined by two non-parallel translations and by only 2-, 3-, 4-, and 6-fold rotations. The glide reflection axis has to be parallel to one of the translation vectors. In one case, two glide reflection axes are perpendicular to each other, both parallel to the two translation vectors. The summarized rules for point, line and plane groups are in the table below:

| operations                                                                                            | point groups                                                                                            | line groups                                                                                                                                                                             | plane groups                                                                                                                                                                                                                                                                                                                                                                             |
|-------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| translation<br>    | $\emptyset$                                                                                             | 1<br>                                                                                                | 2<br>                                                                                                                                                                                                                                                                                                 |
| rotation<br>       | <br>$\alpha = 360/n$ | 180°<br>                                                                                             | 180°<br><br>120°<br><br>90°<br><br>60°<br> |
| reflection<br>     | <br>$\beta = 360/n$  | 90°<br><br>0°<br> | $\alpha$<br><br>30° 45° 60° 90°                                                                                                                                                                                                                                                                       |
| glide reflect.<br> | $\emptyset$                                                                                             | 0°<br>                                                                                               |                                                                                                                                                                                                                                                                                                       |

The point, line and plane groups are presented in a compact, graphic illustration, loaded with information. We use the international terminology, but we replaced the international symbols with our own graphics, aiming to be more user-friendly and intuitive. The content of the fields and the symbols are detailed in the rectangle “Legend”

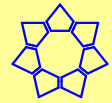
The groups are placed in rows and columns. The rows are headed by the rotation groups (1-, 2-, 3-, 4-, 6-fold), the columns are organized by the other operations. The following data are shown for each group:

- the international label
- the unit cell (red)
- the fundamental region (blue)
- the generators
- the lattice, a regular array of points

The first four items were defined previously. Here, we add further information on the fundamental region. It is either a parallelogram (rectangle, square) or a special hexagon called parallelogon (zonagon): its opposite edges are parallel and of equal lengths. The lattice, or tessellation, is a polygonal grid generated by the symmetry operations, see [PI 2](#).

While [PL 1](#) is crowded with information. It is all visible at once, allows the reader to have an overall look and recognize order and patterns in this presentation.

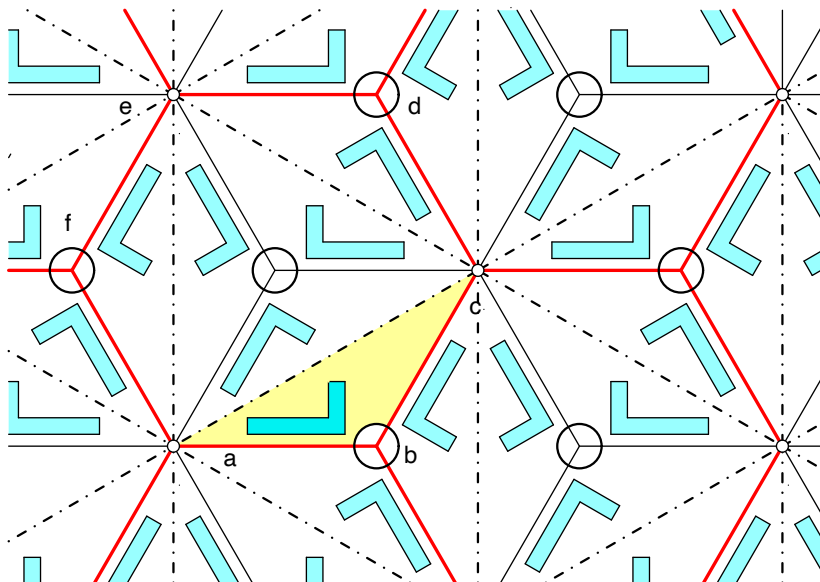
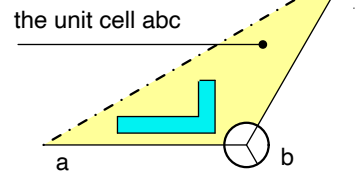
The group may be constructed using only the unit cell, but it is more efficient to use the fundamental region.



To further clarify the concept of a group, we picked **p31m** as an example.

We begin with a 3-fold rotation of the motif around the center  $b$ . The essence of group: while we rotate the motive, the same operation is applied to the other generators, the mirror axis  $ac$ . Now we execute the reflections, which will reproduce the roto-centers. It is interesting to note that you can cover the plane with the motifs without translation. The solid red lines indicate the **fundamental region**  $abcde$ . The point  $B$  is a 3-fold center, the line  $ac$  is a mirror axis.

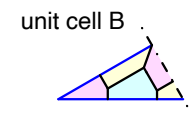
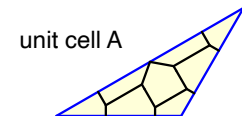
During the multiplication of the motif, the roto-centers and the mirrors are also reflected and rotated.



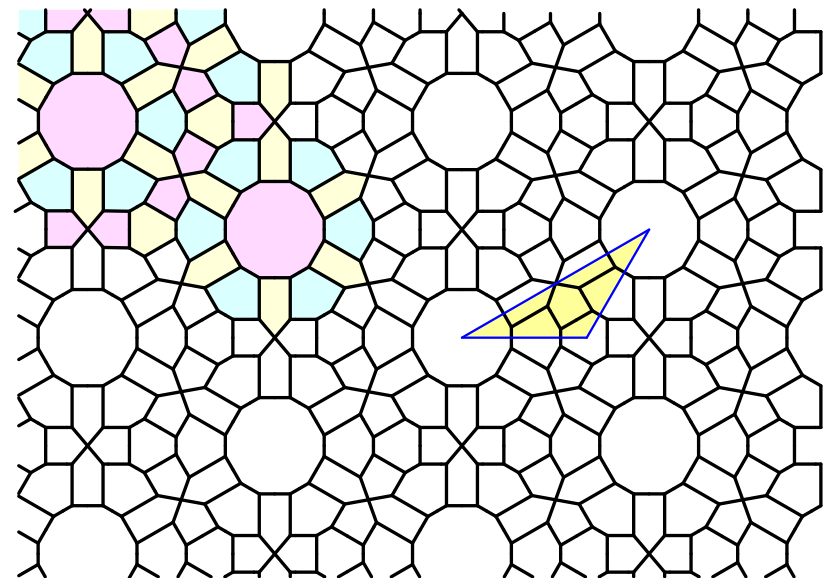
We mentioned earlier the didactic reason for using the letter L as a motif. Here, we try to demonstrate the difference: we use a more complex motif with the same group (**p31m**)

When designing a group (mosaic), it is useful to be aware of the **available choices** (parameters):

- selection of one of the 17 group
- orientation of the unit cell
- scale of the unit cell
- choice of the motif, symmetries

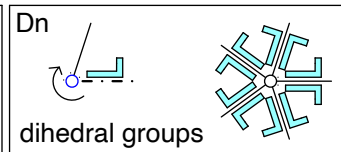
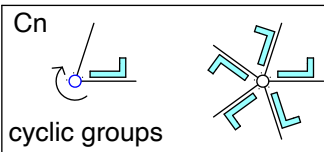


Since the motif has a mirror symmetry, the unit cell B may also be used by applying the operators of the group **p6m**!



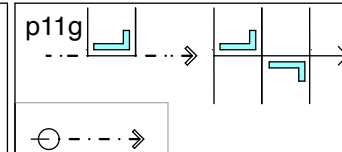
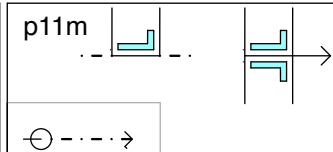
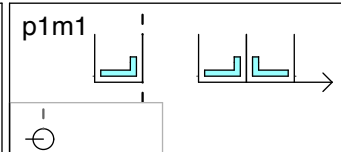
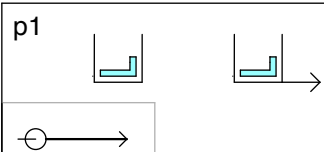
2 x ∞  
Point  
groups

• n=1,2,3...

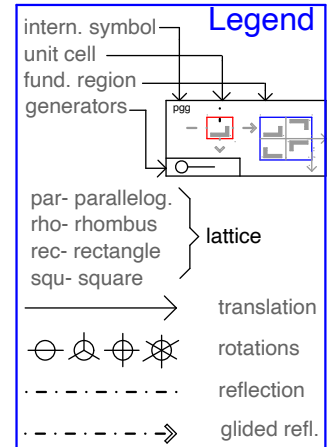
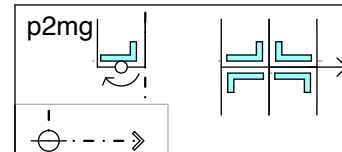
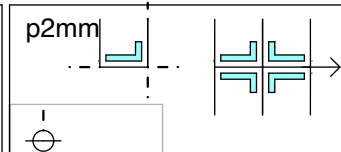
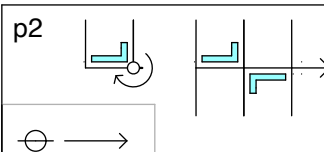


7  
Line  
groups

n=1  
360°

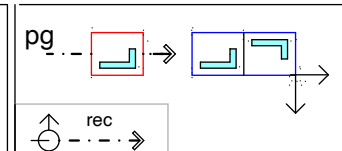
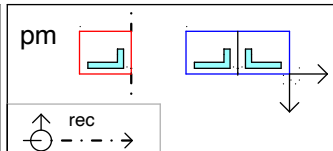
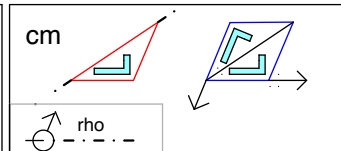
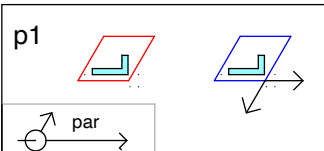


n=2  
180°

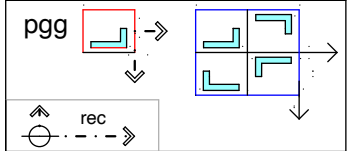
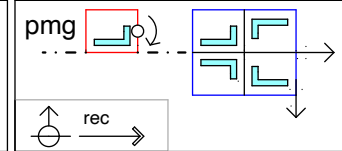
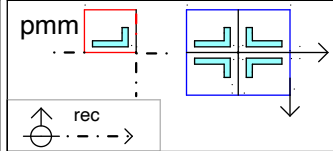
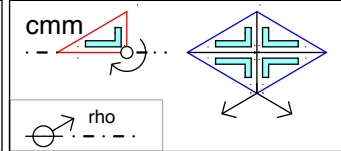
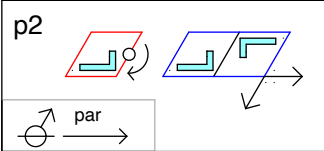


17  
Plane  
groups

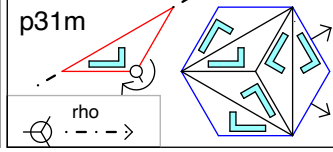
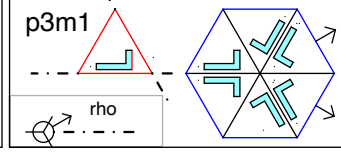
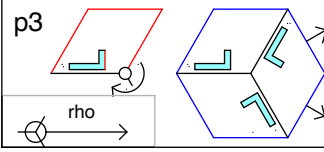
n=1  
360°



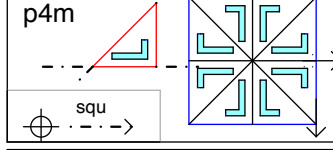
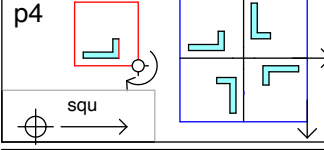
n=2  
180°



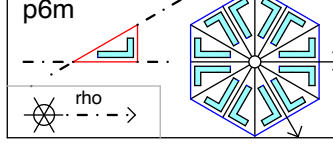
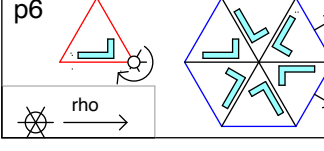
n=3  
120°

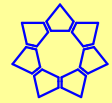


n=4  
90°



n=6  
60°





## 6. The 17 plane groups with lattice PI 2

We use the same tabulated layout as in PI 1: the groups are placed in rows and columns. The rows are headed by the rotation groups (1-, 2-, 3-, 4-, 6-fold), the columns are organized by the other operations.

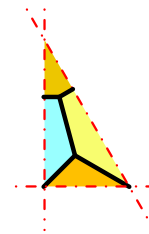
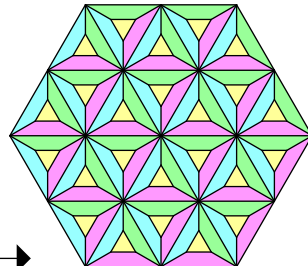
Here, we kept the lattices for the mosaics, revealing the motif's placement relative to the unit cell. This information has a significant impact when designing a group. If the motif partially or entirely lies outside the unit cell, it will result in periodic self-intersections.

## 7. The 17 plane groups, no lattice PI 3

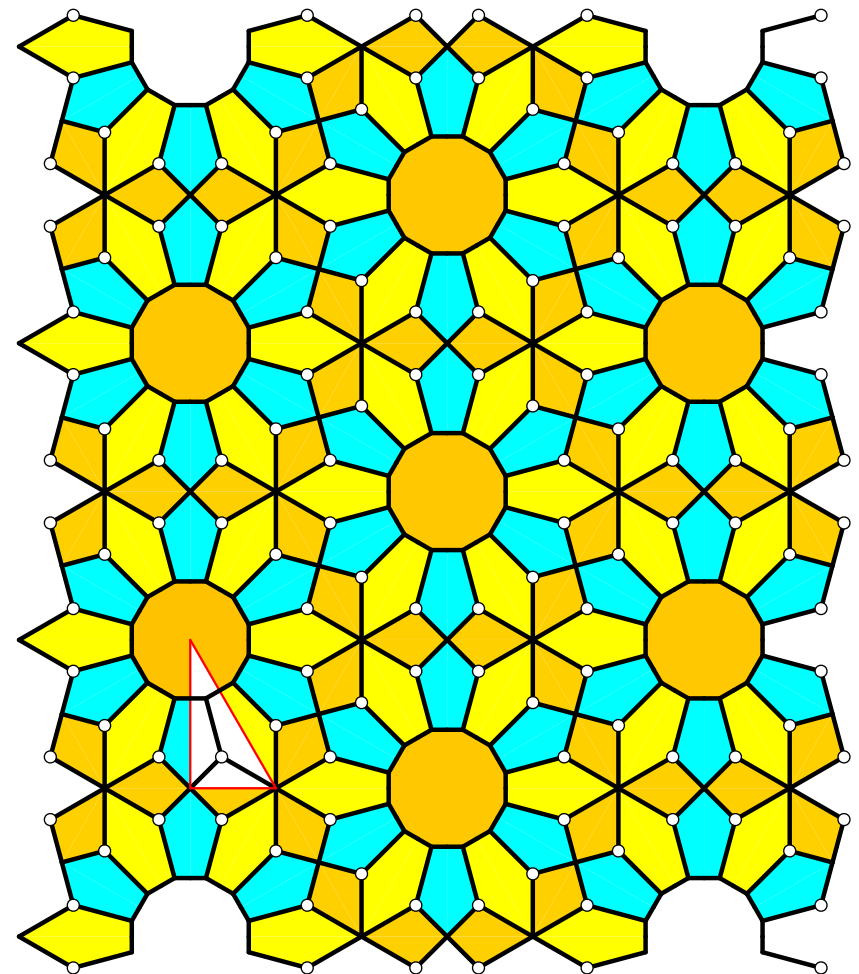
The same illustrations as PI 2, but the lattices are omitted. The design of a mosaic begins with the selection of a group from this table. Viewing all 17 groups on the same page will help distinguish the pattern variations.

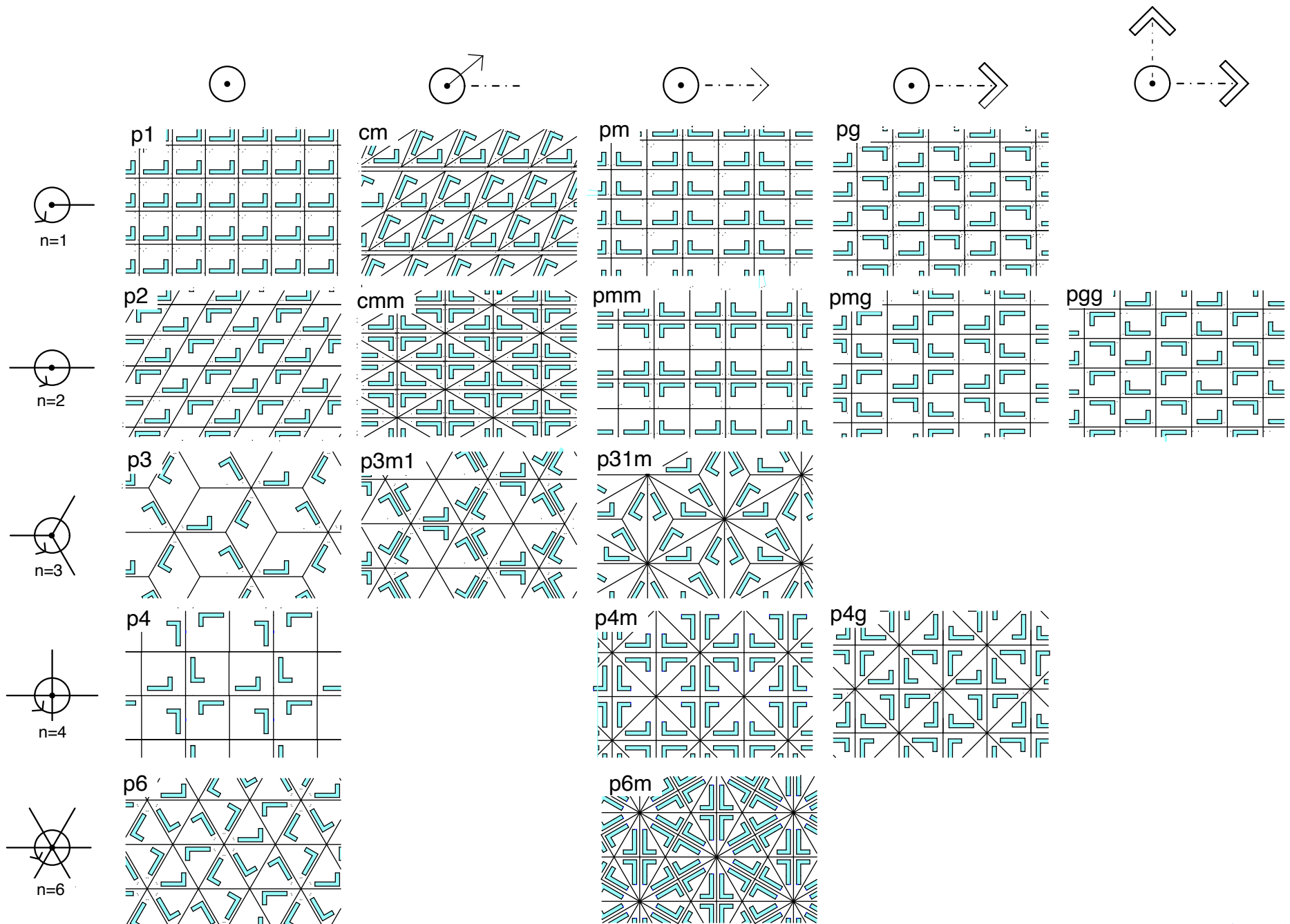
Note:

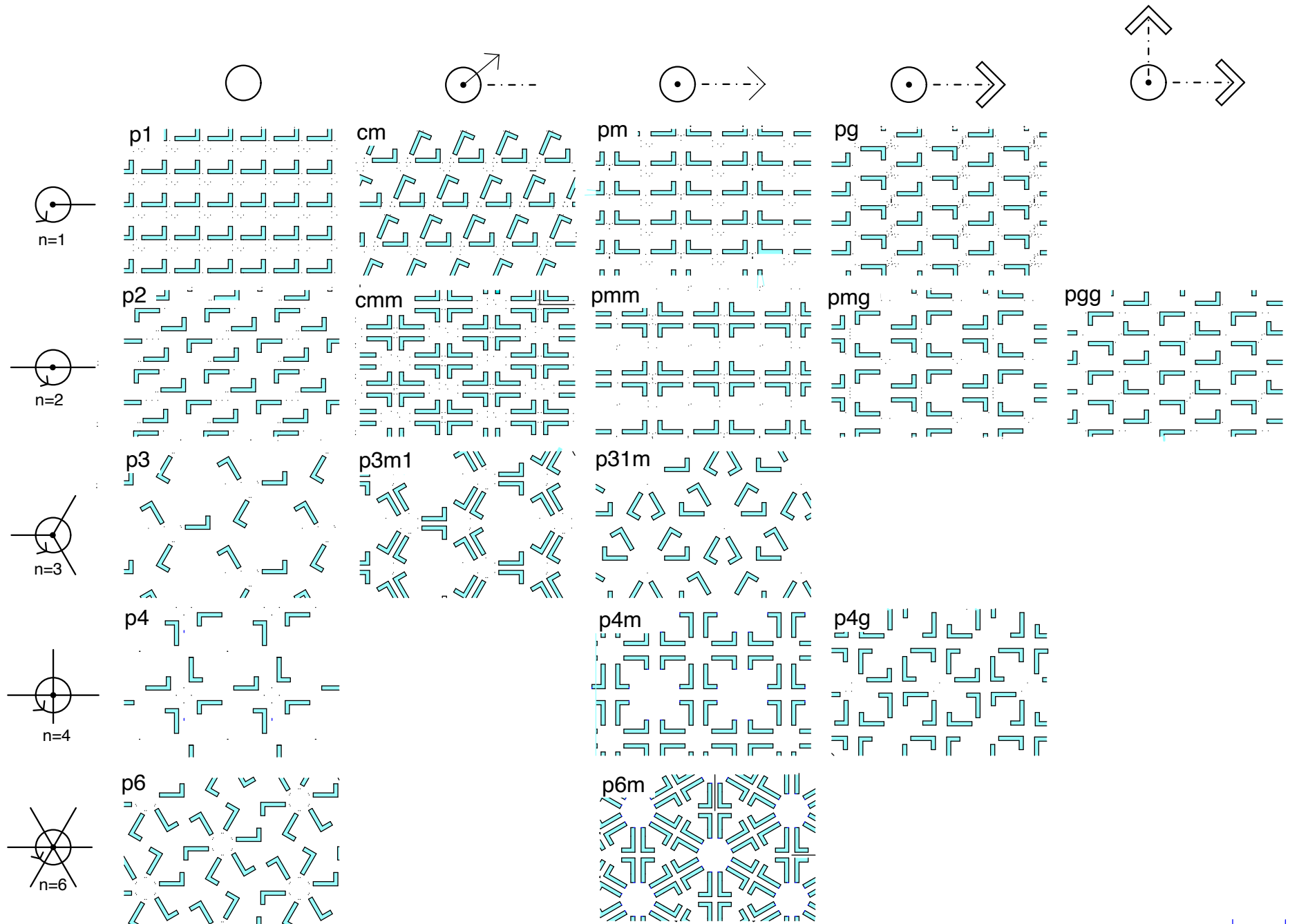
The mathematical theory is developed in the last 60 years, but we use only the simple map colouring principle: adjacent regions should have different colours. Remember: four colours are always sufficient! →

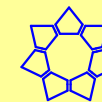


Here is a three colour example: drawing five little segments in a  $30^\circ - 60^\circ$  triangle and multiply it with mirrors on three sides of the triangle by the group  $p6m$ , we receive the pattern below. Without of the digital tools we enjoy today, it is unlikely that we could as quickly arrive at anything similar.









## 8. Tilings - tessellations

PI 4

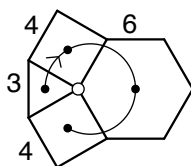
These two terms are interchangeable. Again, the term tiling is rooted in architecture: covering a wall or a floor periodically. Symmetry and regularity are the keywords for tessellations

- Regular tessellation: covering the plane with identical regular polygons without gaps or overlaps
- Semiregular tessellation: covering the plane with more than one type of regular polygon without gaps or overlaps

There are three regular tessellations: the regular (equilateral) triangle, the square and the regular hexagon. For all three cases, the vertices are regular and identical. In a regular vertex, the incident edges form a point group: same angles between adjacent edges, C3, C4, C6.

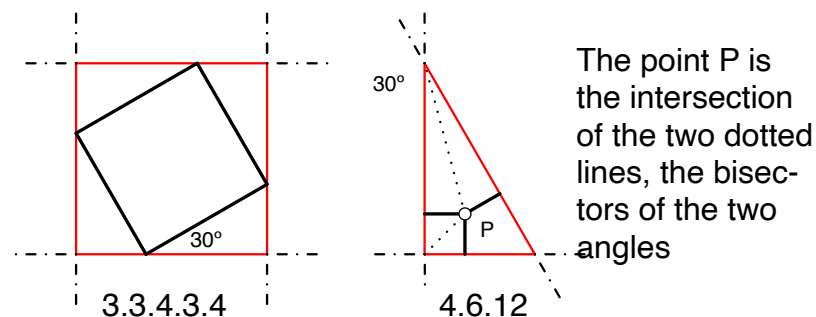
There are eight semiregular tessellations with 3, 4, 6, 8 and 12-sided regular polygons. Here, the vertices are identical, but not regular.

A sequence of cyclic numbers of faces surrounding every vertex defines a tessellation. For instance 3.4.6.4 denotes the faces at every vertex: Since the label is a cyclic permutation, 4.6.4.3 or 6.4.3.4 or 4.3.4.6 are equivalent. To shorten the labels, for instance, 3.3.3.3.6 is denoted by  $3^4.6$  in most publications.

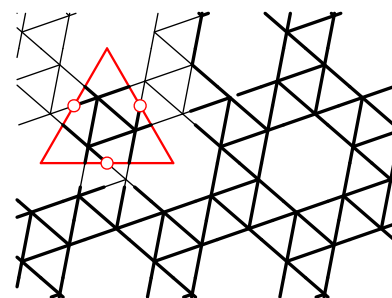


Each of the 11 tessellations is ruled by one of the 17 plane symmetry groups, also shown in PI 4.

The unit cells (defined in 4.) are indicated as a red rectangle (square) or a half-regular triangle ( $30^\circ$ ,  $60^\circ$ ). They are helpful information to draw rapidly the tessellation with a computer by using the edges as mirrors. A few black lines inside the unit cell are sufficient to cover the plane. Two examples:



It is noteworthy that one tessellation, 3.3.3.3.6, is an exception: only rotations, no reflections.



We do have a unit cell here, too, but the operations are 2-fold rotations. The centers of the rotations are the midpoints of the triangle. Two versions exist: they are mirror views or enantiomers.

For the construction of the black lines inside a regular triangle, see PI 6.

Legend:

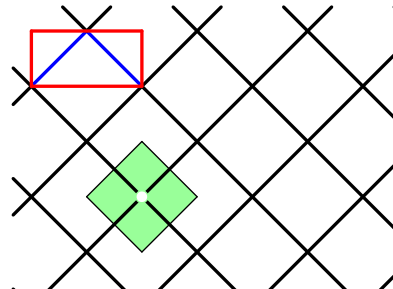
Unit cell



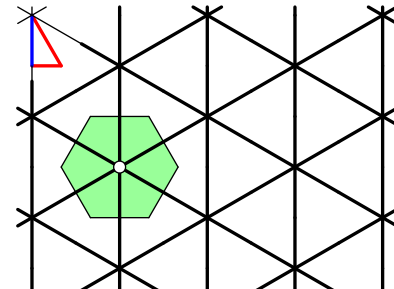
Dual tile



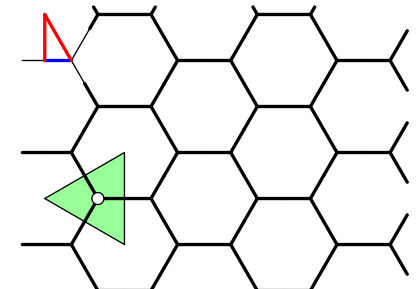
incidence symbol, plane group



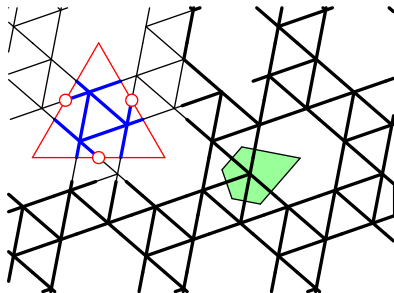
4.4.4.4 • p4m • 4<sup>3</sup>



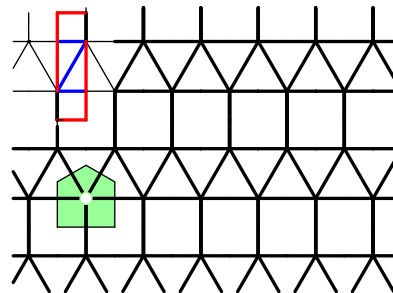
3.3.3.3.3.3 • 6pm • 3<sup>6</sup>



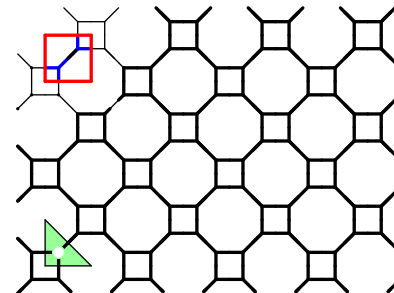
6.6.6 • p6m • 6<sup>3</sup>



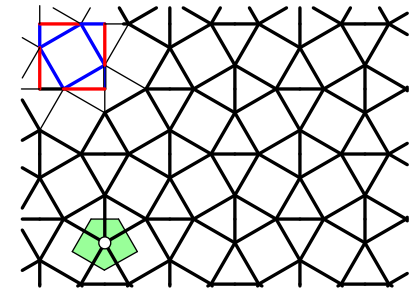
3.3.3.3.6 • p6 • 3<sup>4</sup>.6



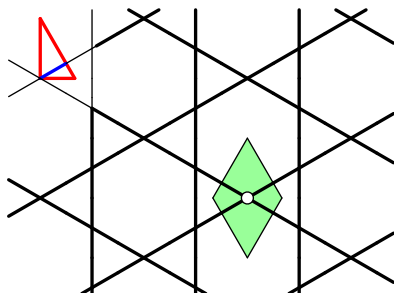
3.3.3.4.4 • cmm • 3<sup>3</sup>.4<sup>2</sup>



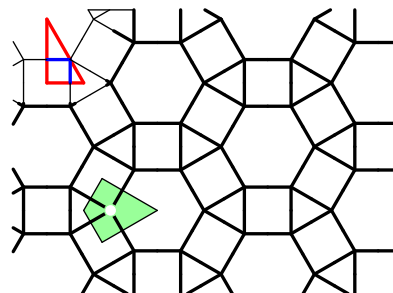
4.8.8 • p4m • 4.8<sup>2</sup>



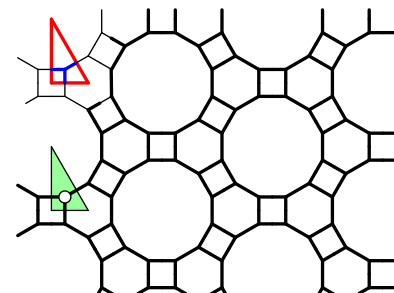
3.3.4.3.4 • p4g • 3<sup>2</sup>.4.3.4



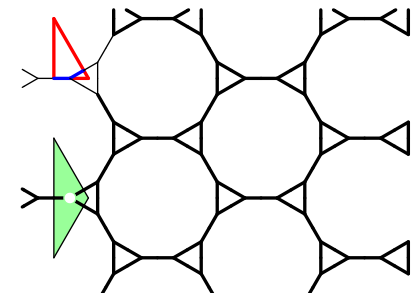
3.6.3.6 • p6m



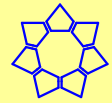
3.4.6.4 • p6m



4.6.12 • 6pm



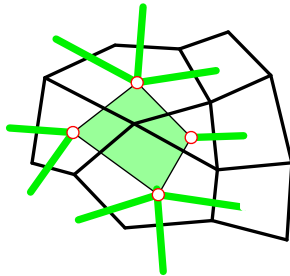
3.12.12 • p6m • 3.12<sup>2</sup>



## 9. Dual Tiling

PI 5

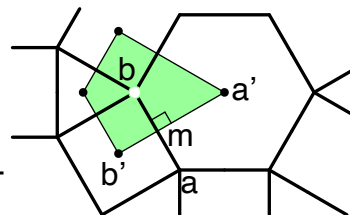
A dual tessellation is created in the plane by replacing every face (tile) by a vertex and joining a pair of vertices if their corresponding faces have a common edge. The result is that every face is replaced by a vertex, every vertex is replaced by a face, and a new edge replaces every original edge. Every  $n$ -gon is replaced by an  $n$ -valent vertex and every  $n$ -valent vertex by an  $n$ -valent polygon:



The new dual face of a regular or semiregular tessellations is shown as a green-coloured polygon on PI 4.

The metric construction in the case of regular and semiregular tessellations are simple and rapid: draw the bisectors of the edges, they intersect in the dual tessellation:

The vertex  $a'$  is the center of the regular hexagon, the vertex  $b'$  is the center of the triangle. The new edge  $a'b'$  is perpendicular to the old edge  $ab$  and passes by the midpoint  $m$  of  $ab$



The duals of the three regular tessellations are regular tessellations them-self. The square tessellation is self-dual, and the triangular and the hexagonal are duals of each other. The duals of the 8 semiregular tessellations give birth to new faces. The vertices are regular, the faces are non-regular. The table below exposes these interesting properties:

|          | regular tessellation | semiregular tessellation | dual tessellation |
|----------|----------------------|--------------------------|-------------------|
| vertices | regular              | non-regular              | regular           |
| faces    | regular              | regular                  | non-regular       |

It is evident that a particular tessellation and its dual belong to the same symmetry group.

## 10. Summary of tilings PI 6

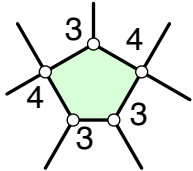
PI 6 depicts all the graphic information about the semiregular tessellations.

The following non-regular convex polygons cover the plane periodically:

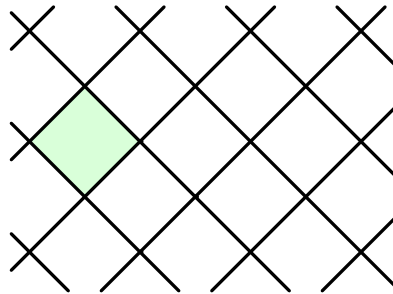
1. any triangle
2. any quadrilateral
3. 15 families of pentagons
4. 3 families of hexagons

Convex polygons with more than 6 sides do not cover the plane.

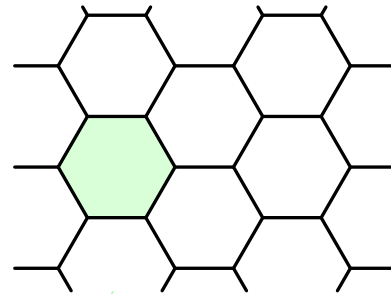
Symbole: 3.3.4.3.4



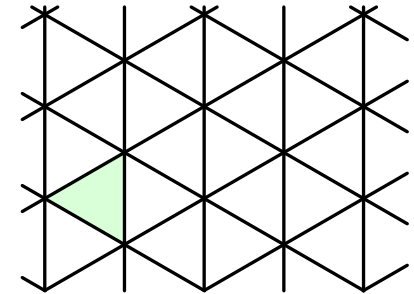
incidence symbol, plane group



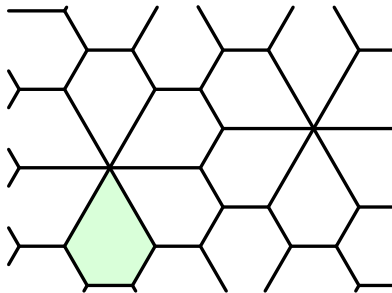
4.4.4.4 • p4m • 4<sup>3</sup>



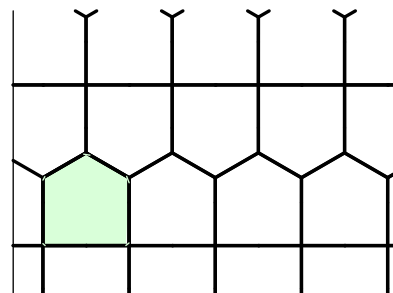
3.3.3.3.3.3 • 6pm • 3<sup>6</sup>



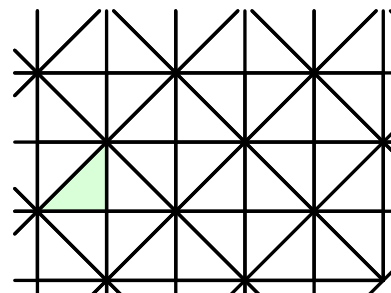
6.6.6 • p6m • 6<sup>3</sup>



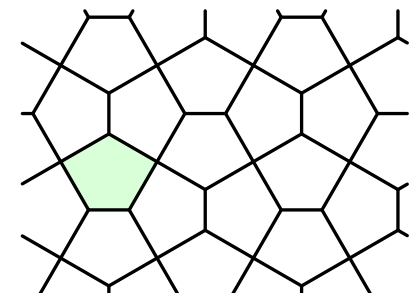
3.3.3.3.6 • p6



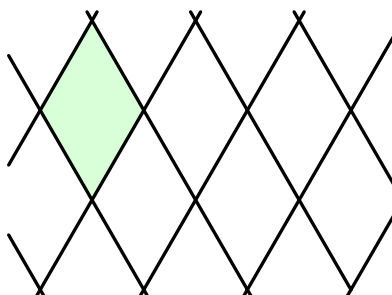
3.3.3.4.4 • cmm • 3<sup>3</sup>.4<sup>2</sup>



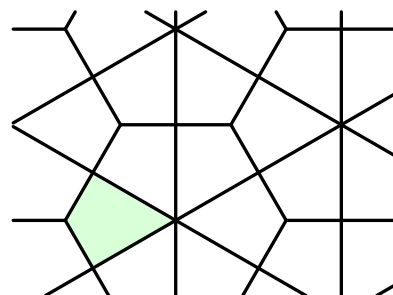
4.8.8 • p4m • 4.8<sup>2</sup>



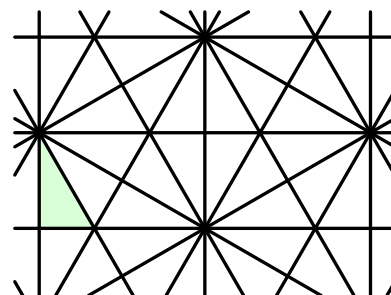
3.3.4.3.4 • p4g • 3<sup>2</sup>.4.3.4



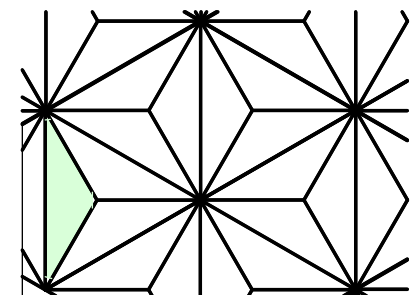
3.6.3.6 • p6m



3.4.6.4 • p6m

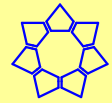


4.6.12 • 6pm



3.12.12 • p6m • 3.12<sup>2</sup>

|              | 3.3.3.3.6<br>p6      | 3.3.3.4.4<br>cmm | 4.8.8<br>p4m | 3.3.4.3.4<br>p4g | 3.6.3.6<br>p6m       | 3.4.6.4<br>p6m       | 4.6.12<br>p6m        | 3.12.12<br>p6m       |
|--------------|----------------------|------------------|--------------|------------------|----------------------|----------------------|----------------------|----------------------|
| Incidence    |                      |                  |              |                  |                      |                      |                      |                      |
| unit cell    | <br>regular triangle | <br>rectangle    | <br>square   | <br>square       | <br>30°-60° triangle | <br>30°-60° triangle | <br>30°-60° triangle | <br>30°-60° triangle |
| fund. region | <br>hexagon          | <br>hexagon      | <br>hexagon  | <br>square       | <br>parallelogram    | <br>hexagon          | <br>hexagon          | <br>hexagon          |
| dual tile    | <br>60°              | <br>120°         | <br>45°      | <br>120°         | <br>120°             | <br>120°             | <br>30°              | <br>120°             |



## 11. Isohedral tiles

PI 7

Isohedral tiles are convex\* polygons that cover the plane with identical copies without gaps or overlaps, arranged by a symmetry group. A finite number of existing tiles may create infinite designs by modifying certain edges (tilings), adding a motif (patterns), or both.

The graphic artist M. C. Escher (1898-1972) explored this field with spectacular success before the complete list of 93 tiles were first published in 1987 by the mathematicians Grünbaum/Sheppard. We present the 43 non-symmetrical tiles on PI 7, and the 38 symmetrical tiles on plates PI 21. We omitted the 12 “marked” tiles; they don’t offer visually distinct designs.

The 43 tiles are depicted in a tabulated form on PI 7, with an explanation of our graphic symbols. We present two examples of tilings for each case by modifying certain edges on plates PI 9 to PI 20.

We grouped the tiles by the number of their edges. The letter in the label (T, Q, P, H) refers to triangle, quadrilateral, pentagon and hexagon. They are numbered in increasing order by the parametric number, **Pn**. This innovative concept is a very helpful tool for constructing the polygon precisely. Pn will be explained in the following.

\* In a convex polygon, all interior angles are less than 180 degrees.

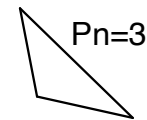
## 12. Parametric number Pn

*Pn: the number of parameters (data) necessary and sufficient to define a plane configuration. The parameters are distances and angles, they should be independent and compatible.*

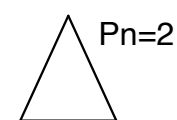
The simplest example: a triangle is defined by the lengths of its 3 edges, or 2 edges and an angle, or 1 edge and 2 angles. If 3 angles are given, an infinity of triangles can be drawn, but the triangle is not defined: the third angle is dependent on the other two

$$\gamma = (180^\circ - \alpha - \beta)$$

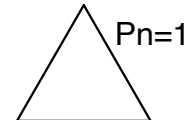
thus, we have only 2 independent parameters. In a triangle, the length of any edge should be less than the sum of the two others; otherwise, the 3 lengths are incompatible. If we impose symmetry or regularity, Pn will be less than 3.



any triangle



isosceles triangle

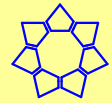


equilateral triangle

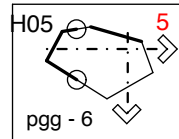
The terms symmetry and regularity present additional hidden information, i.e. equal lengths or angles.

The formula for any plane polygon with V vertices:

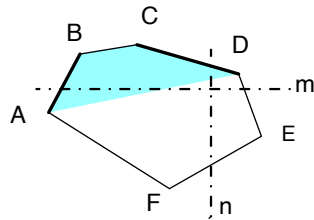
$$Pn = 2V - 3$$



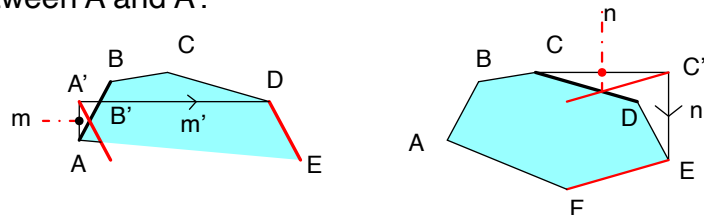
To further clarify the concept of  $P_n$ , we analyze a more complex case, for instance, the tile H05.



H05 is a hexagon; thus, in general position,  $P_n = 2V - 3 = 9$ . However, the two glide reflections impose restrictions, and  $P_n$  becomes 5. Here is the explication: the first four vertices are freely chosen,  $2 \times 4 - 3 = 5$



The edge DE is now uniquely determined: the edge AB is reflected about the horizontal axis m and glided vertex A' into vertex D. The axis m is the bisector of the distance between A and A'.



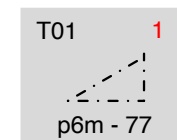
Similarly, the edge CD is glide reflected into the edge EF. Now, the hexagon is entirely defined. The first 4 vertices A, B, C and D were freely chosen, and E and F were constructed.

Please note that m and n have to be perpendicular; m' and m, n' and n are parallel pairwise.

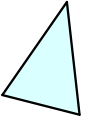
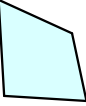
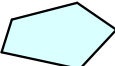
It is essential to point out the choices while creating a tiling.

1. The selection of the tile from [PI 7](#) may be based on the number of edges of the polygon and (or) based on the preference of the symmetry group.
2. Selection based on the parametric number: higher numbers mean more options and greater freedom to design a tiling
3. Scale of the polygon: it will determine the density of the tiling.
4. Orientation of the polygon: the direction of one of the translations may be parallel to the X-axis (our preference in the following) or oblique (any angle).
5. Modification of the edges: the strongest visual impact on the tiling (see pages 24 and 26).

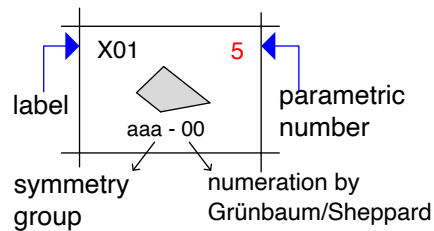
One triangular tile is greyed out on [PI 7](#): T01. The edges of this tile are not modifiable because they are the axis of reflection (mirrors).



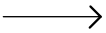
We will see three more tiles with a similar structure among the symmetrical tiles ([PI 21](#)), whose edges are all mirrors. However, these tiles may also result in visually attractive designs by combining them with motifs (see [PI 22](#) and [PI 23](#))

|                                                                                     |                              |                               |                              |                               |                              |                              |                              |                              |                              |
|-------------------------------------------------------------------------------------|------------------------------|-------------------------------|------------------------------|-------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|
|    | T01 <sup>1</sup><br>p6m - 77 | T02 <sup>1</sup><br>p31m - 38 | T03 <sup>1</sup><br>p6 - 39  | T04 <sup>1</sup><br>p4g - 81  | T05 <sup>1</sup><br>P4 - 79  | T06 <sup>1</sup><br>p6 - 88  | T07 <sup>2</sup><br>cm - 83  | T08 <sup>2</sup><br>pgg - 86 | T09 <sup>2</sup><br>cmm - 78 |
|                                                                                     | T10 <sup>3</sup><br>p2 - 84  | T11 <sup>3</sup><br>pmg - 85  |                              |                               |                              |                              |                              |                              |                              |
|    | Q01 <sup>1</sup><br>p3 - 33  | Q02 <sup>1</sup><br>p4g - 56  | Q03 <sup>1</sup><br>p6 - 31  | Q04 <sup>2</sup><br>p31m - 30 | Q05 <sup>2</sup><br>p4g - 56 | Q06 <sup>2</sup><br>pgg - 52 | Q07 <sup>3</sup><br>cmm - 54 | Q08 <sup>3</sup><br>p1 - 41  | Q09 <sup>3</sup><br>cm - 45  |
|                                                                                     | Q10 <sup>3</sup><br>pmg - 50 | Q11 <sup>3</sup><br>pg - 44   | Q12 <sup>3</sup><br>pm - 42  | Q13 <sup>3</sup><br>p2 - 47   | Q14 <sup>3</sup><br>pg - 43  | Q15 <sup>3</sup><br>pgg - 51 | Q16 <sup>4</sup><br>pgg - 53 | Q17 <sup>4</sup><br>pmg - 49 | Q18 <sup>5</sup><br>p2 - 46  |
|    | P01 <sup>3</sup><br>p4 - 28  | P02 <sup>3</sup><br>p6 - 21   | P03 <sup>3</sup><br>pgg - 27 | P04 <sup>4</sup><br>pgg - 25  | P05 <sup>4</sup><br>cm - 22  | P06 <sup>5</sup><br>p2 - 23  | P07 <sup>5</sup><br>pmg - 24 |                              |                              |
|  | H01 <sup>3</sup><br>p3 - 7   | H02 <sup>5</sup><br>pg - 2    | H03 <sup>5</sup><br>pg - 3   | H04 <sup>5</sup><br>p1 - 1    | H05 <sup>5</sup><br>pgg - 6  | H06 <sup>6</sup><br>pgg - 5  | H07 <sup>7</sup><br>p2 - 4   |                              |                              |

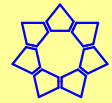
## Legend



rotations: 2-fold  180°  
 3-fold  120°  
 4-fold  90°  
 6-fold  60°

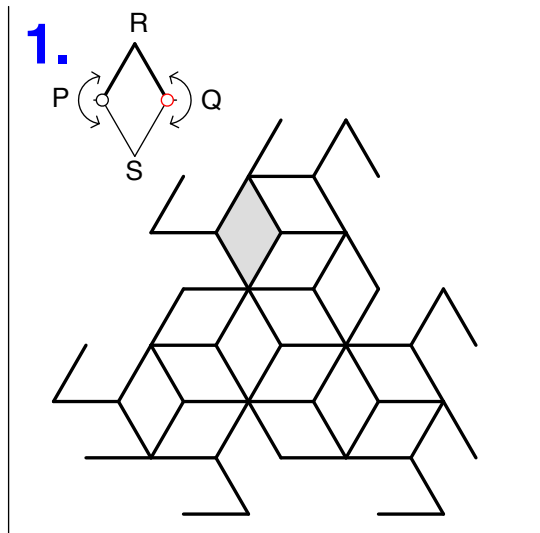
translation   
 reflection   
 glided reflection 

modifiable edge   
 derived edge   
 modifiable half-edge 



## 13. Modifying the edges

As an example, we show a simple modification of the edges of the tile Q01 (Fig. 1.) The parametric number ( $P_n=1$ ) tells you that the choice of a single length is sufficient to draw the polygon ( $60^\circ$  rhombus, 3-fold rotations). Repeated 3-fold rotations around the centres P and Q results in the well-known tiling below.

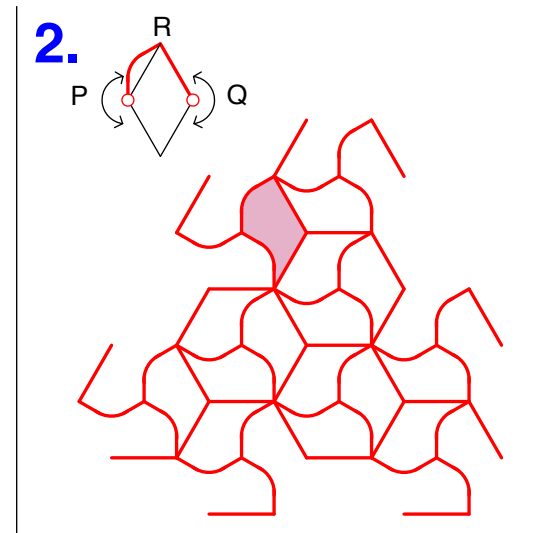


While the modifications are free to choose, we may want to avoid intersecting edges.

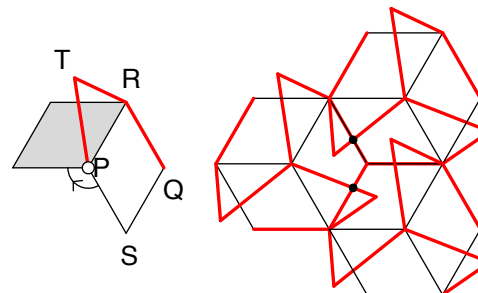
If only one edge is modified (PR) and it is not entirely within the grey parallelogram, it will result in intersecting edges (Fig. 4). The original tile is rotated  $120^\circ$  around the vertex P to obtain the grey parallelogram.

In Fig. 5, we modified the other available edge RQ. Here, the two modified edges intersect. To test this type of intersection, rotate the modified edge PR  $120^\circ$  around P.

The two edges indicated with thick lines (PR and RQ) can be modified. In Fig. 2. the edge PR is replaced by an arbitrary curve; the new polygon will be rotated and will result in a new tiling. The edges PS and SQ are copies of the edges PR and RQ.

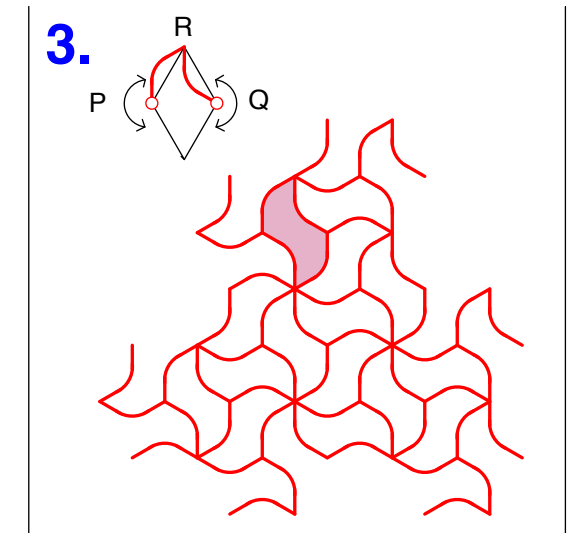


4. The vertex T is situated outside the grey parallelogram, the edge TR intersect the edge RQ

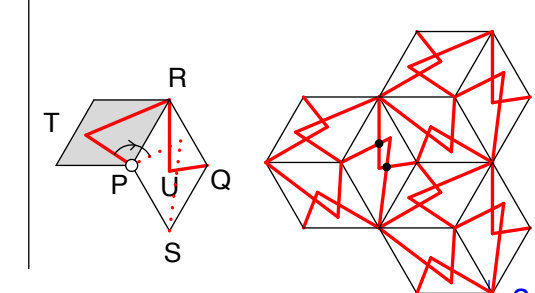


In Fig. 3. the edge RQ is also modified. If you compare the tilings in Fig. 1 with Fig. 3, you may realize how this simple act of the edge modification offers an infinite variety of the creative process. Escher used this method and created adroit, interlocking living creatures:

*"how did he do it?"*

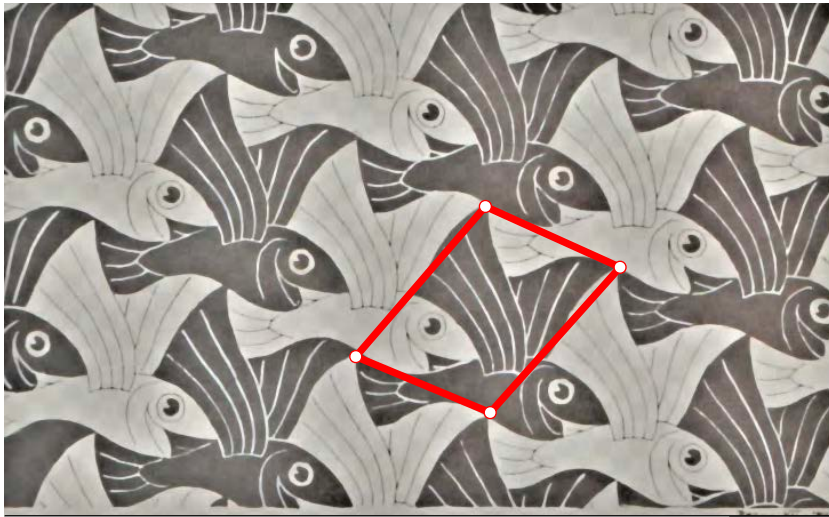


5. The vertex T is situated inside the grey parallelogram, but it intersects with the modified edge RU

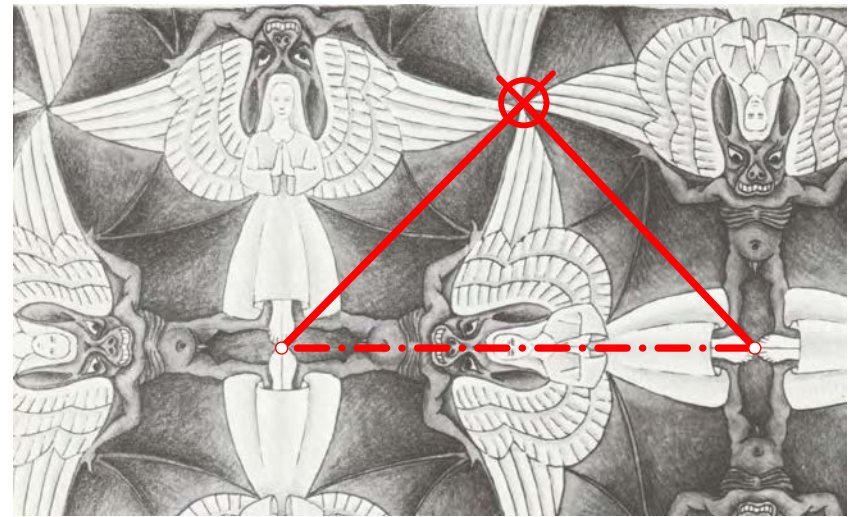


• **Translation** • Birds • 1938

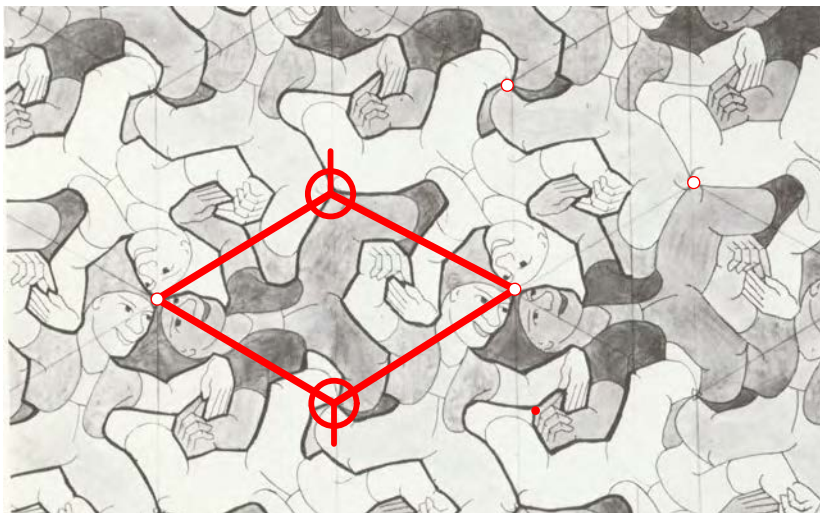
Q08, p1



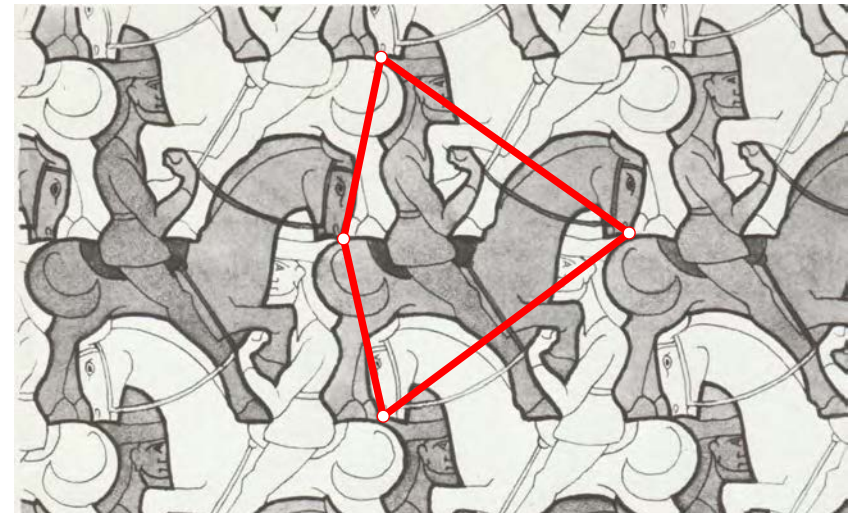
• **Reflection** • Angels and devils • 1941 • T04, p4g

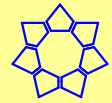


• **Rotation** • Symmetry Work 21 • 1938 • Q01, p3

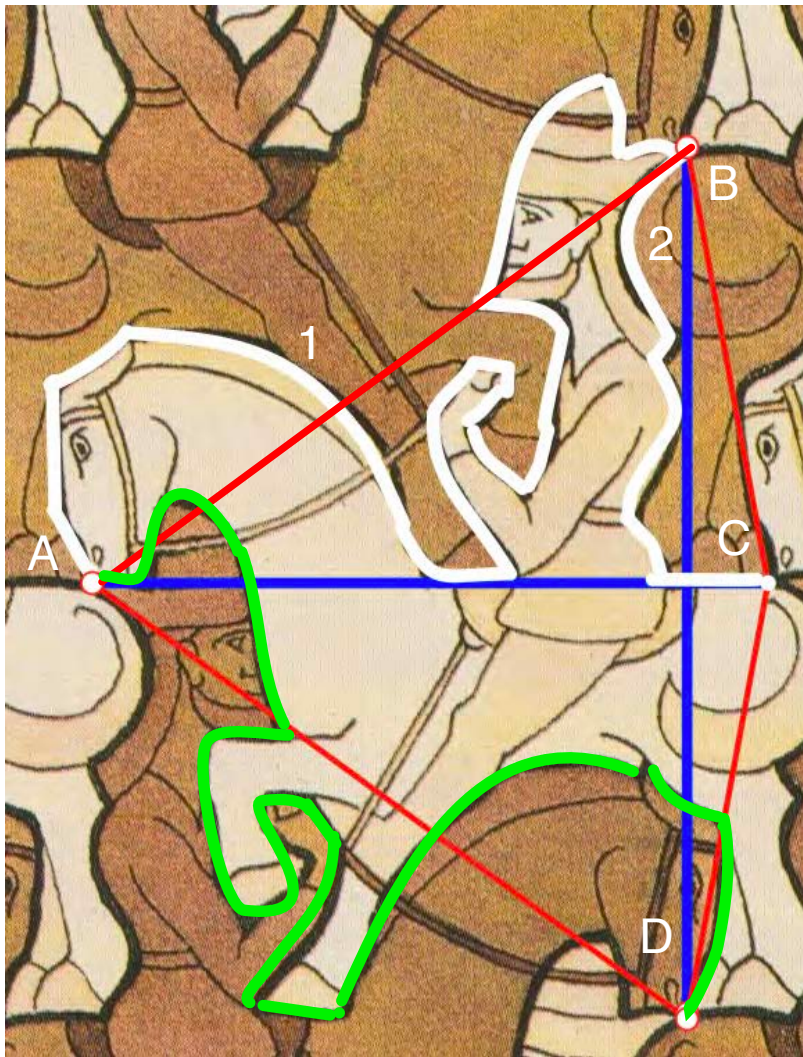


• **Glide reflection** • The Horsman • 1946 • Q11, pg



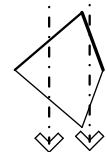
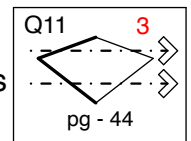


## 14. M. C. Escher: The Horseman 1946

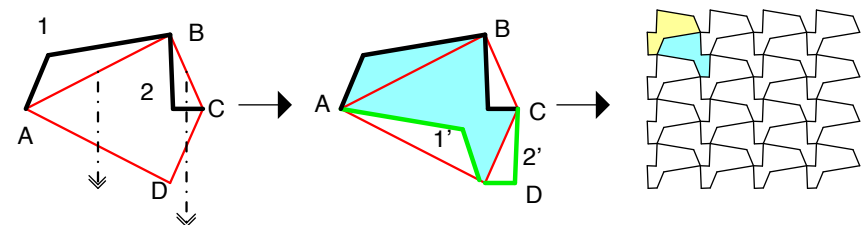


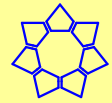
The Horseman by the graphic artist M. C. Escher is analyzed on the left. He used the quadrilateral tile Q12 from [PI 7](#). In 1946, this information was not available; Grünbaum/Shephard published them for the first time in 1987.

The tile is subjected to two parallel glide reflections, and two edges are modifiable. The parametric number is 3, thus the vertices A, B and C are freely chosen, and B is reflected into D by the axis AC. This tile belongs to the group **pg**. We mentioned earlier the available choices during design, among them orientation. Here, the tile is rotated 90° in relation to our model; thus, the glide reflection vectors are “vertical” (parallel to the Y-axis). The edge AB is modified into path 1. The edge BC is independently modified into path 2. Note that the white path is reflected and translated into the green path.

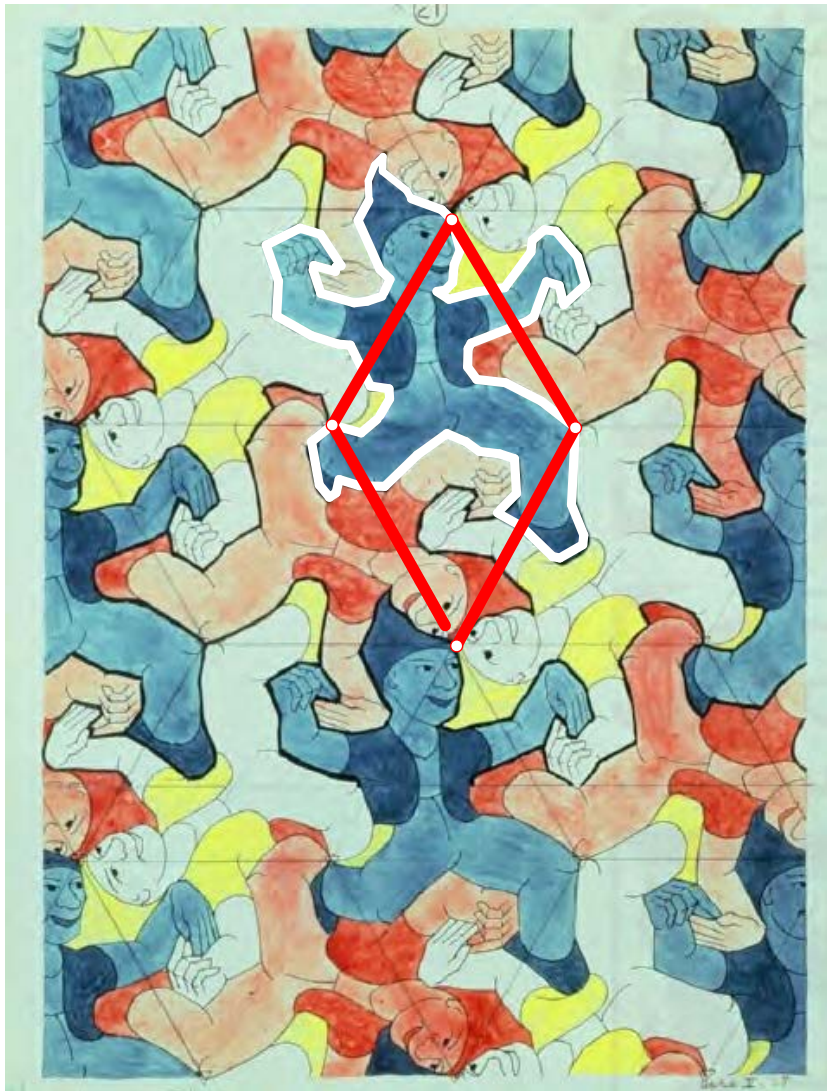


Besides the geometric ingenuity of Escher, the creation of the two modified edges so that the brown and the beige horses (they are identical) meet without gaps or overlaps reveals an unusual, unique imagination. Here is a very simple illustration of the modifications:

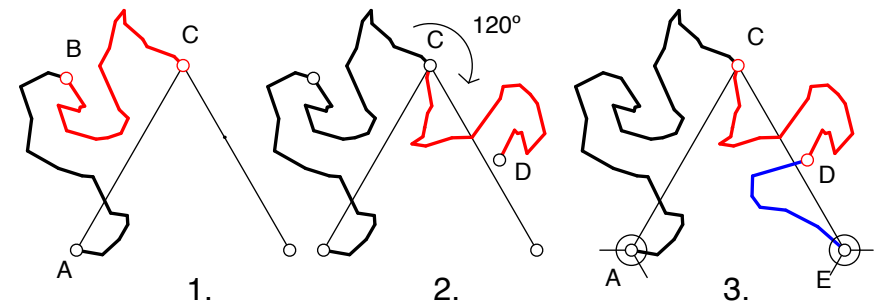
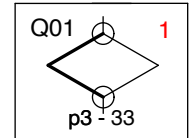




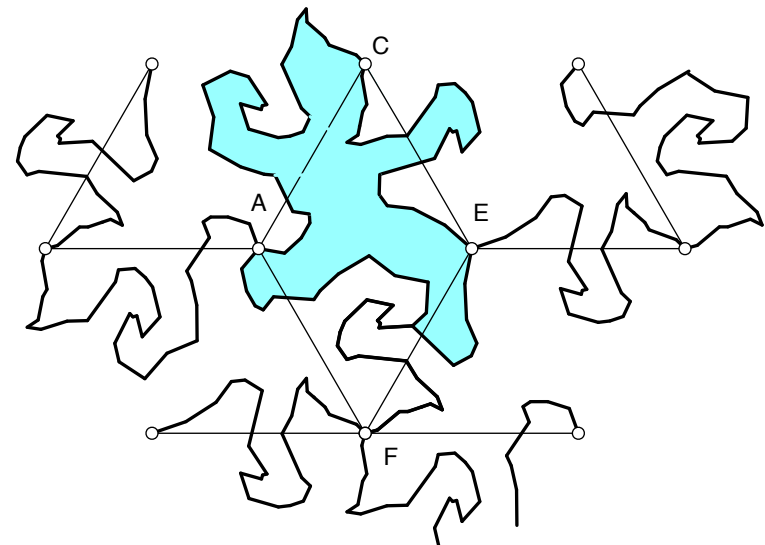
## 15. M. C. Escher: Symmetry Work 21

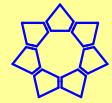


Symmetry Work 21 was a study (1938) to prepare Cycle, a Woodcut. He used the tile Q01 with a clever twist. The edge AC is modified into the path ABC, but the partial path BC (?) is rotated  $120^\circ$  instead of freely modifying CE. The partial path DE is “freely” chosen.



Now we 3-fold rotate the path ABC around the center A and the path CDE around center E; figure ACEF appears.

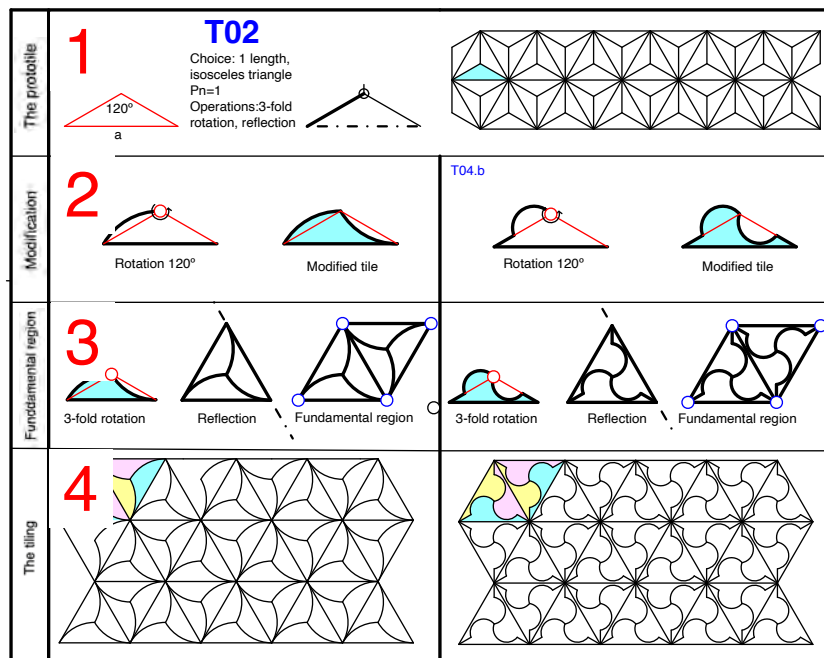




## 16. Tiling samples PI 9 to PI 20

The following plates present two tiling samples for 42 of the 43 tiles of PI 7 (T01 edges are not modifiable). Although PI 7 contains enough data to construct any of the 42 tilings, we included these samples; they give you a visual comparison of the available design options.

The sequence of plates follows the order of PI 7: triangular, quadrilateral, pentagonal and hexagonal tiles. The data is presented in four sections:



1. The **Prototile**: we used this term for a set of two diagrams. The first is the geometrical representation of the tile, and the second shows the symbols of the symmetry operations necessary to tile the plane. On the right, the tessellation is shown before modifying the edges.
2. The **Modification**: this is the process where the brilliance of Escher is unmatched, but even the uninitiated can achieve pleasing effects. For the sake of uniformity and clarity, we choose the most straightforward edge modifications:

• translated edge •



• 2-fold rotated edge •

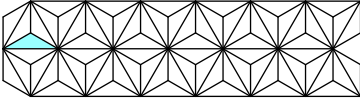
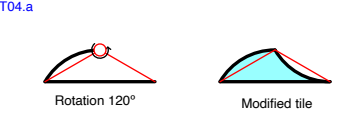
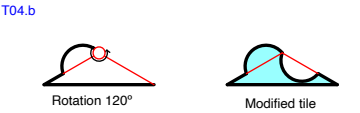
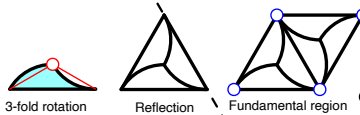
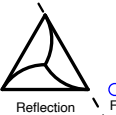
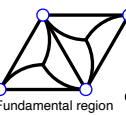
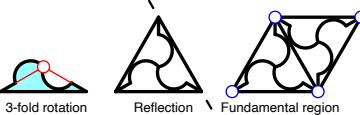
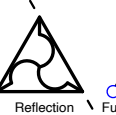
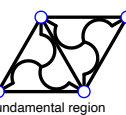
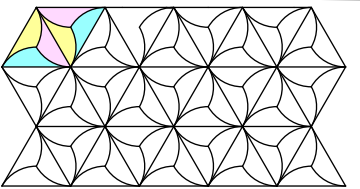
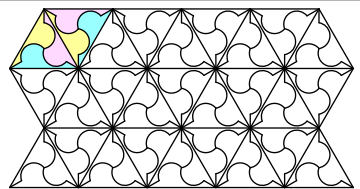


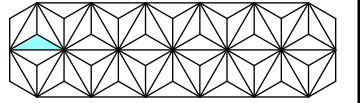
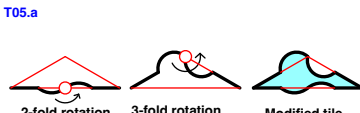
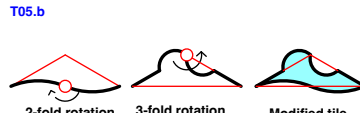
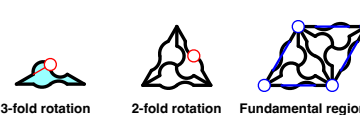
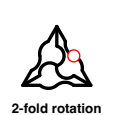
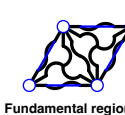
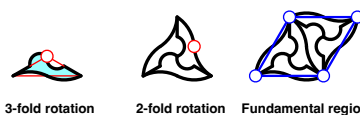
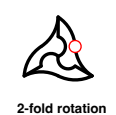
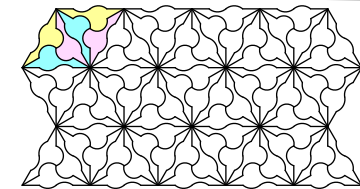
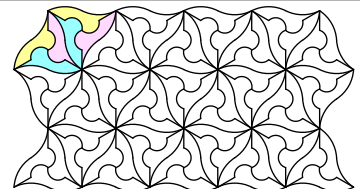
• glide reflected edge •

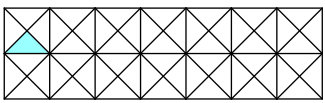
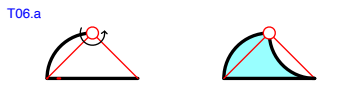
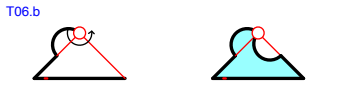
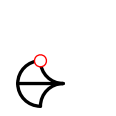
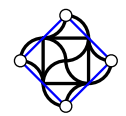
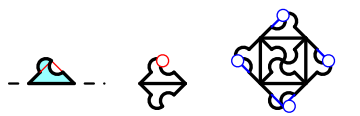
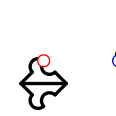
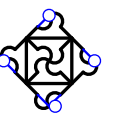
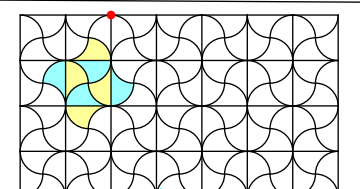
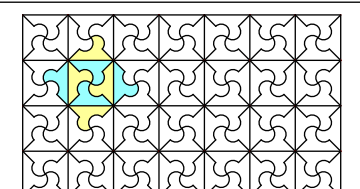


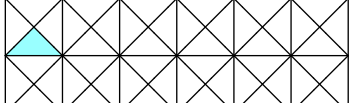
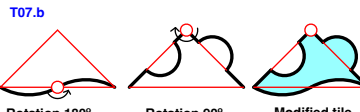
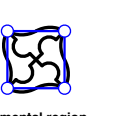
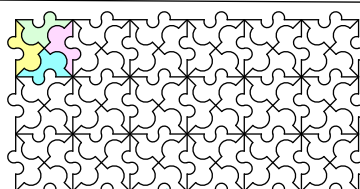
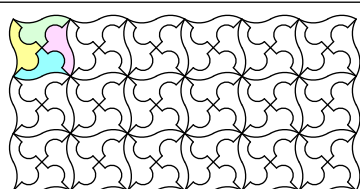
3. The **Fundamental region**: defined on p.6 and p.7

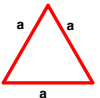
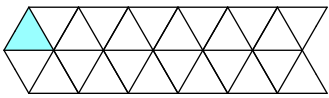
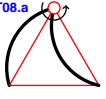
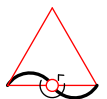
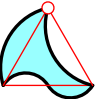
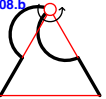
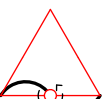
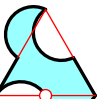
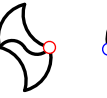
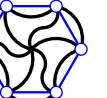
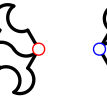
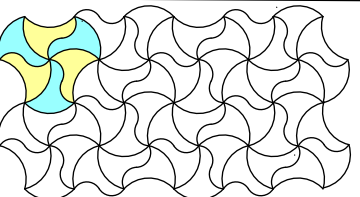
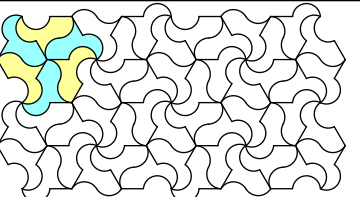
4. The **Tiling**: we coloured the fundamental region. If you translate it in two directions, our condition is satisfied: no two adjacent regions have the same colours. On the right side of the plates, we show the result of an alternate modification. You may create new tilings by removing equivalent edges in a periodical, rhythmic fashion. Enjoy!

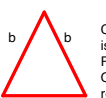
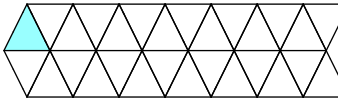
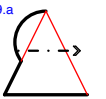
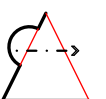
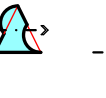
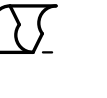
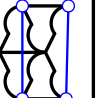
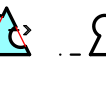
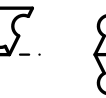
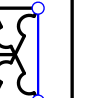
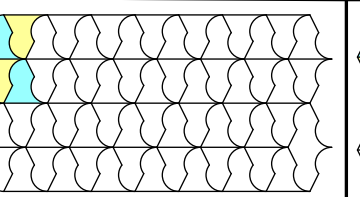
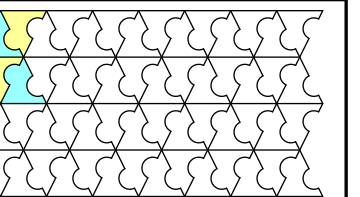
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| The prototype      | <b>T02</b><br>Choice: 1 length, isosceles triangle<br>$P_n=1$<br>Operations: 3-fold rotation, reflection                                                                                                                                                                                                  |                                                                                                                                                                                                                                                                                                             |  |
| Modification       | <b>T04.a</b><br> Rotation 120°<br> Modified tile                                                                                        | <b>T04.b</b><br> Rotation 120°<br> Modified tile                                                                                        |                                                                                    |
| Fundamental region |  3-fold rotation<br> Reflection<br> Fundamental region |  3-fold rotation<br> Reflection<br> Fundamental region |                                                                                    |
| The tiling         |                                                                                                                                                                                                                          |                                                                                                                                                                                                                           |                                                                                    |

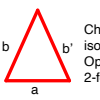
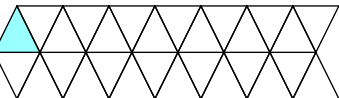
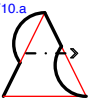
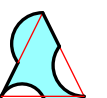
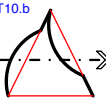
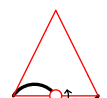
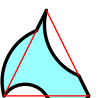
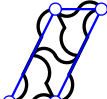
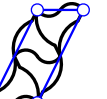
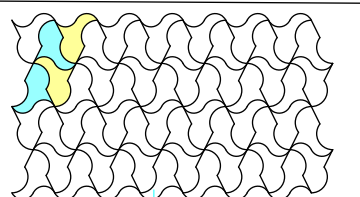
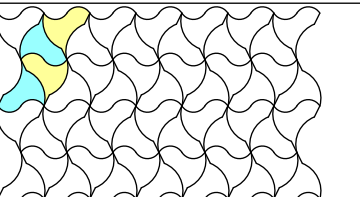
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| The prototype      | <b>T03</b><br>Choice: 1 length, isosceles triangle<br>$P_n=1$<br>Operations: 2-fold rotation, 1 3-fold rotation                                                                                                                                                                                                      |                                                                                                                                                                                                                                                                                                                      |  |
| Modification       | <b>T05.a</b><br> 2-fold rotation<br> Modified tile                                                                                             | <b>T05.b</b><br> 2-fold rotation<br> Modified tile                                                                                             |                                                                                     |
| Fundamental region |  3-fold rotation<br> 2-fold rotation<br> Fundamental region |  3-fold rotation<br> 2-fold rotation<br> Fundamental region |                                                                                     |
| The tiling         |                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                   |                                                                                     |

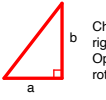
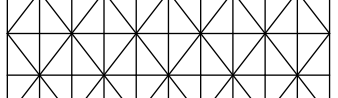
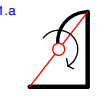
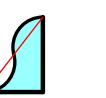
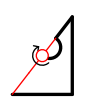
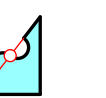
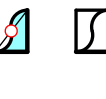
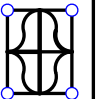
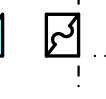
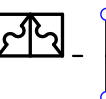
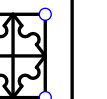
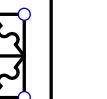
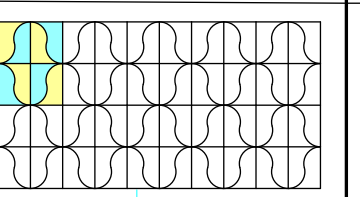
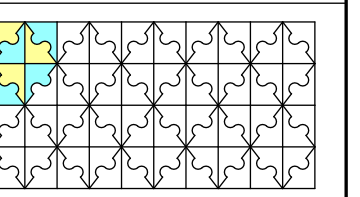
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|--------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
| The prototype      | <b>T04</b><br>Choice: 1 length, isosceles triangle<br>$P_n=1$<br>Operations: 4-fold rotation, reflection                                                                                                                                                                                                        |                                                                                                                                                                                                                                                                                                                   |  |
| Modification       | <b>T06.a</b><br> Rotation 90°<br> Modified tile                                                                                           | <b>T06.b</b><br> Rotation 90°<br> Modified tile                                                                                           |                                                                                     |
| Fundamental region |  Reflection<br> 4-fold rotation<br> Fundamental region |  Reflection<br> 4-fold rotation<br> Fundamental region |                                                                                     |
| The tiling         |                                                                                                                                                                                                                              |                                                                                                                                                                                                                               |                                                                                     |

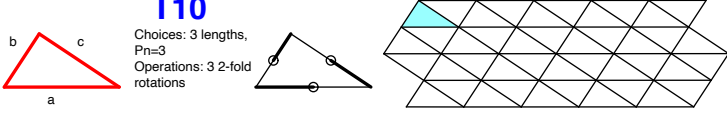
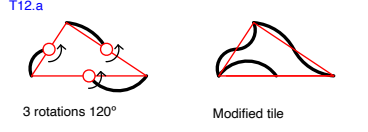
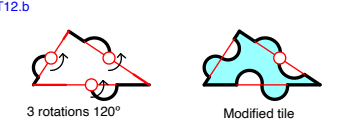
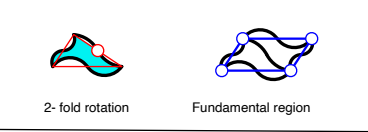
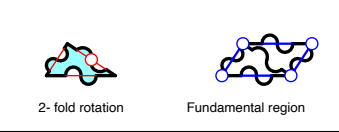
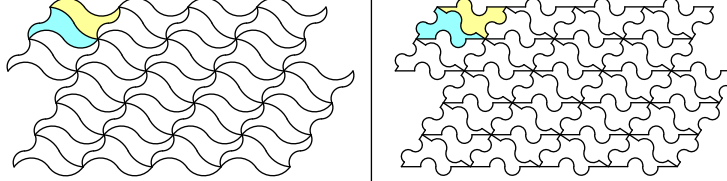
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| The prototype      | <b>T05</b><br>Choice: 1 length, isosceles triangle<br>$P_n=1$<br>Operations: 2-fold and 4-fold rotation                                                                                                                    |                                                                                                                                                                                                                           |  |
| Modification       | <b>T07.a</b><br> Rotation 180°<br> Modified tile | <b>T07.b</b><br> Rotation 90°<br> Modified tile |                                                                                      |
| Fundamental region |  4-fold rotation<br> Fundamental region          |  4-fold rotation<br> Fundamental region         |                                                                                      |
| The tiling         |                                                                                                                                       |                                                                                                                                      |                                                                                      |

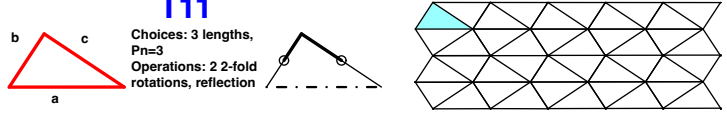
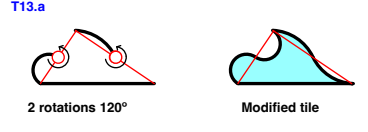
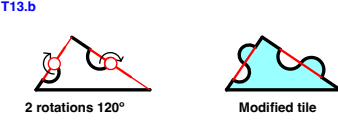
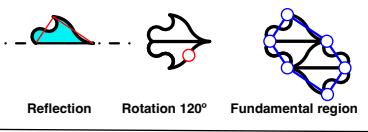
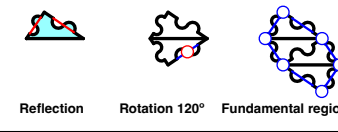
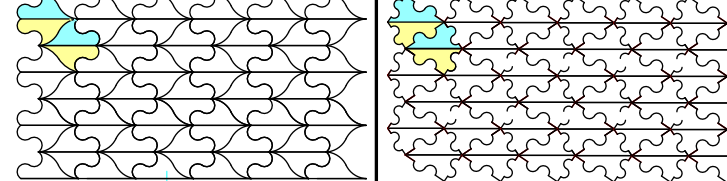
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| The prototype      | <b>T06</b><br> Choice: 1 length, equilateral triangle<br>$Pn=1$<br>Operations: 6-fold and 2-fold rotations |                                                                                                   |                                                                                                      |                              |                                                                                                   |                                                                                                       |
| Modification       | <b>T08.a</b><br> Rotation 60°                                                                              |  Rotation 180°   |  Modified tile      | <b>T08.b</b><br> Rotation 60° |  Rotation 180°   |  Modified tile      |
| Fundamental region |  2-fold rotation                                                                                           |  3-fold rotation |  Fundamental region |  2-fold rotation              |  3-fold rotation |  Fundamental region |
| The tiling         |                                                                                                            |                                                                                                   |                                                                                                      |                              |                                                                                                   |                                                                                                       |

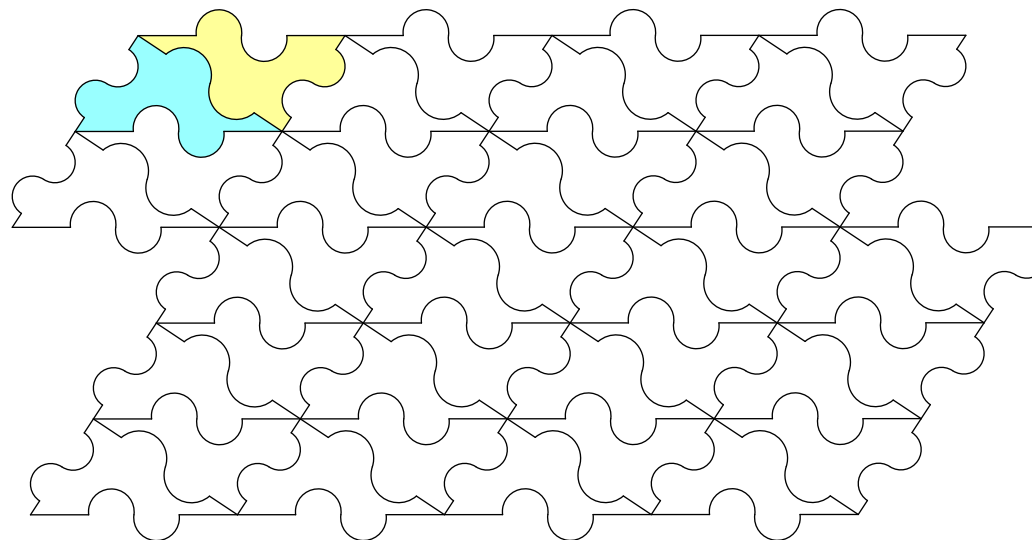
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| The prototype      | <b>T07</b><br> Choice: 2 lengths, isosceles triangle<br>$Pn=2$<br>Operations: glide reflection, reflection |                                                                                                   |                                                                                                                      |                   |                                                                                                |                                                                                                        |
| Modification       | <b>T09.a</b><br> Glide reflection                                                                          |  Modified tile | <b>T09.b</b><br> Glide reflection |  Modified tile    |                                                                                                |                                                                                                        |
| Fundamental region |  Glide reflection                                                                                          |  Reflection    |  Fundamental region               |  Glide reflection |  Reflection |  Fundamental region |
| The tiling         |                                                                                                            |                                                                                                   |                                                                                                                      |                   |                                                                                                |                                                                                                        |

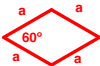
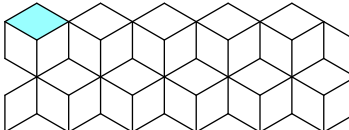
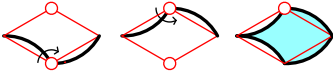
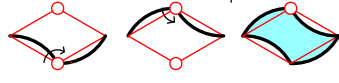
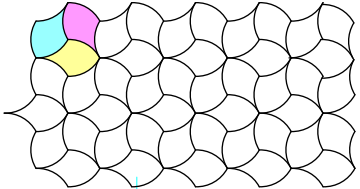
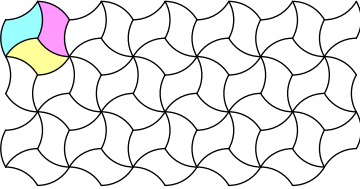
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| The prototype      | <b>T08</b><br> Choices: 2 lengths $Pn=2$ , isosceles triangle<br>Operations: glide reflection, 2-fold rotation |                                                                                                   |                                                                                                        |                                    |                                                                                                   |                                                                                                         |
| Modification       | <b>T10.a</b><br> Glide reflection                                                                             |  Rotation 180° |  Modified tile      | <b>T10.b</b><br> Glide reflection |  Rotation 180° |  Modified tile      |
| Fundamental region |  Glide reflection                                                                                             |  Rotation 180° |  Fundamental region |  Glide reflection                 |  Rotation 180° |  Fundamental region |
| The tiling         |                                                                                                               |                                                                                                   |                                                                                                        |                                  |                                                                                                   |                                                                                                         |

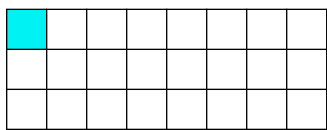
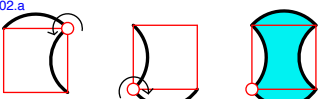
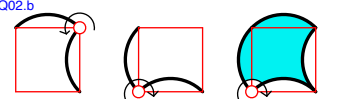
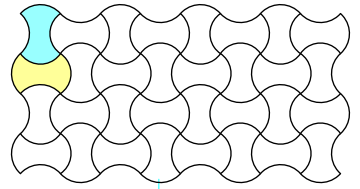
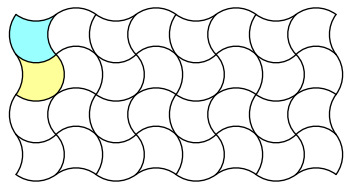
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| The prototype      | <b>T09</b><br> Choices: 2 lengths, right triangle $Pn=2$<br>Operations: 2-fold rotation, 2 reflections |                                                                                                     |                                                                                                                     |                    |                                                                                                  |                                                                                                  |                                                                                                          |
| Modification       | <b>T11.a</b><br> Rotation 120°                                                                        |  Modified tile | <b>T11.b</b><br> Rotation 120° |  Modified tile   |                                                                                                  |                                                                                                  |                                                                                                          |
| Fundamental region |  2-fold rotation                                                                                      |  Reflection    |  Fundamental region            |  2-fold rotation |  Reflection |  Reflection |  Fundamental region |
| The tiling         |                                                                                                       |                                                                                                     |                                                                                                                     |                  |                                                                                                  |                                                                                                  |                                                                                                          |

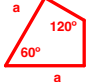
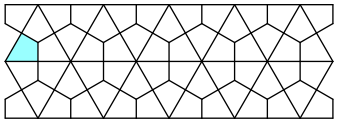
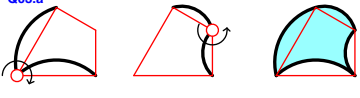
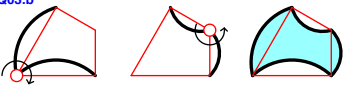
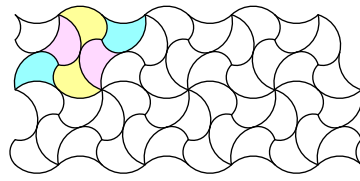
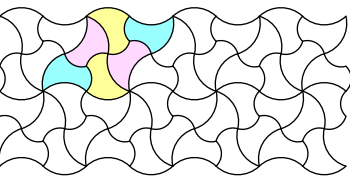
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| The prototype      | <b>T10</b><br>Choices: 3 lengths, $Pn=3$<br>Operations: 3 2-fold rotations  |                                                                                                    |
| Modification       | <b>T12.a</b><br>                                                             | <b>T12.b</b><br> |
| Fundamental region |                                                                              |                  |
| The tiling         |                                                                             |                                                                                                    |

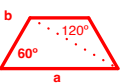
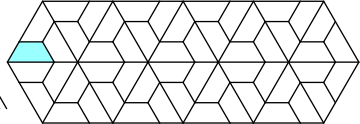
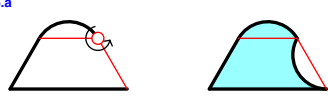
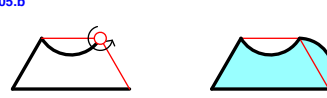
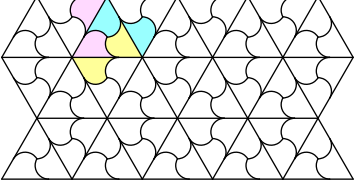
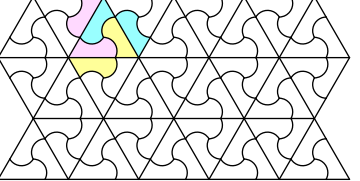
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| The prototype      | <b>T11</b><br>Choices: 3 lengths, $Pn=3$<br>Operations: 2 2-fold rotations, reflection  |                                                                                                     |
| Modification       | <b>T13.a</b><br>                                                                        | <b>T13.b</b><br> |
| Fundamental region |                                                                                         |                  |
| The tiling         |                                                                                         |                                                                                                     |

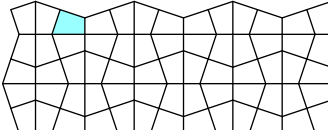
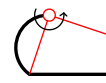
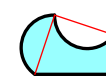
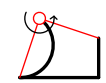
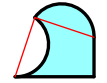
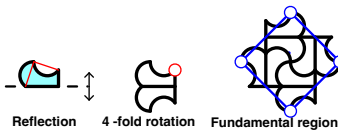
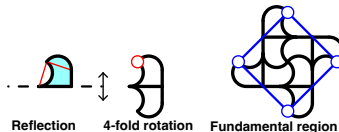
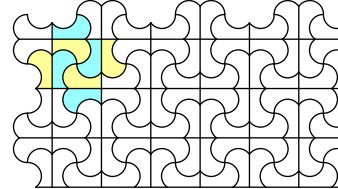
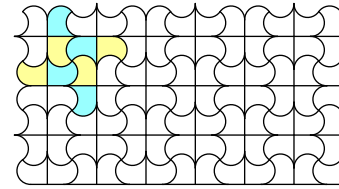


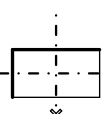
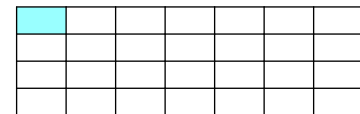
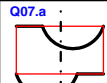
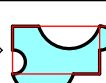
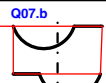
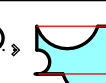
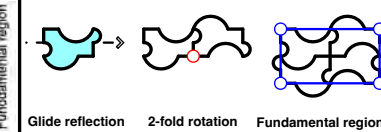
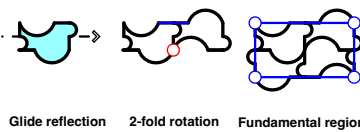
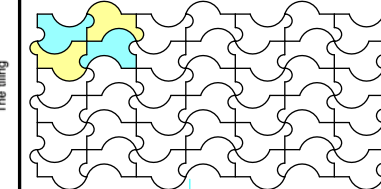
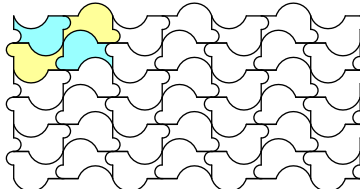
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| The prototype      | <b>Q01</b><br> <p>Choice: 1 length, <math>P_n=1</math><br/>Operations: 2 3-fold rotations</p>  |                                                                                                                                                                                                                       |
| Modification       | <b>Q01.a</b><br> <p>Rotations 120°   Rotations 120°   Modified tile</p>                                                                                                          | <b>Q01.b</b><br> <p>Rotations 120°   Rotations 120°   Modified tile</p>                                                             |
| Fundamental region |  <p>3 fold rotation</p>  <p>Fundamental region</p>                                              |  <p>3 fold rotation</p>  <p>Fundamental region</p> |
| The tiling         |                                                                                                |                                                                                                                                                                                                                       |

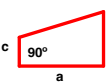
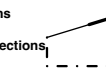
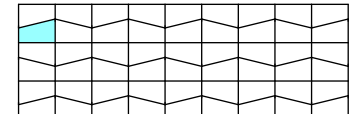
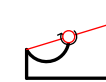
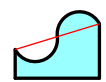
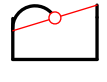
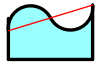
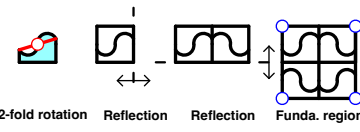
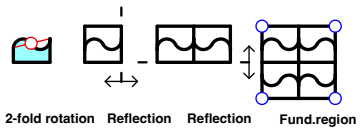
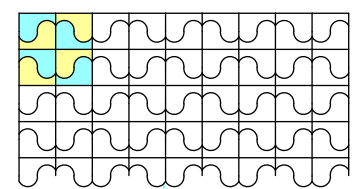
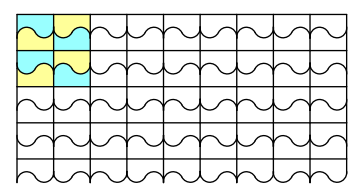
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| The prototype      | <b>Q02</b><br> <p>Choice: 1 length, <math>P_n=1</math><br/>Operations: 2 4-fold rotations</p>  |                                                                                                                                                                                                                       |
| Modification       | <b>Q02.a</b><br> <p>Rotation 90°   Rotation 90°   Modified tile</p>                                                                                                               | <b>Q02.b</b><br> <p>Rotation 90°   Rotation 90°   Modified tile</p>                                                                |
| Fundamental region |  <p>Rotation 90°</p>  <p>Fundamental region</p>                                                |  <p>Rotation 90°</p>  <p>Fundamental region</p> |
| The tiling         |                                                                                                |                                                                                                                                                                                                                       |

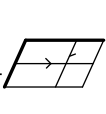
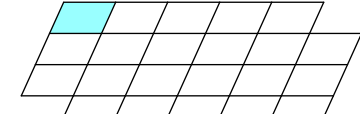
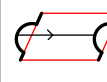
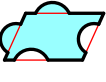
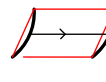
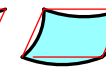
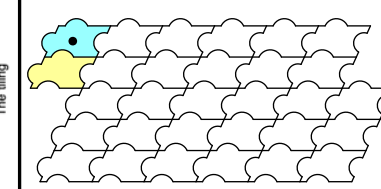
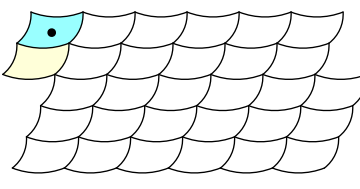
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| The prototype      | <b>Q03</b><br> <p>Choice: 1 length, <math>P_n=1</math><br/>Operations: 3-fold and 6-fold rotations</p>                                                          |                                                                                                                                                                                                                                                                                                                                      |
| Modification       | <b>Q03.a</b><br> <p>Rotation 60°   Rotation 120°   Modified tile</p>                                                                                                                                                                             | <b>Q03.b</b><br> <p>Rotation 60°   Rotation 120°   Modified tile</p>                                                                                                                                                                             |
| Fundamental region |  <p>3 fold rotation</p>  <p>2 fold rotation</p>  <p>Fundamental region</p> |  <p>3 fold rotation</p>  <p>2 fold rotation</p>  <p>Fundamental region</p> |
| The tiling         |                                                                                                                                                              |                                                                                                                                                                                                                                                                                                                                      |

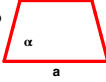
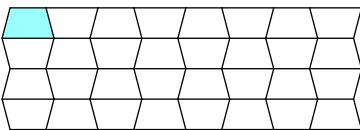
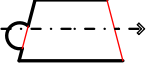
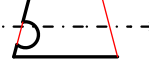
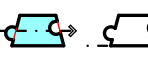
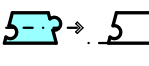
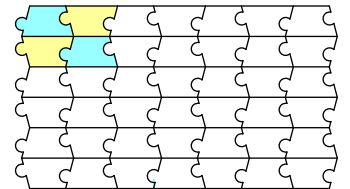
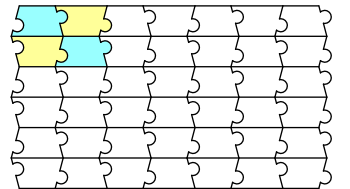
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| The prototype      | <b>Q04</b><br> <p>Choices: 2 lengths, <math>a</math> and <math>b</math>, <math>P_n=2</math><br/>Operations: 3-fold rotation and 2 reflections</p>            |                                                                                                                                                                                                                                                                                                                                         |
| Modification       | <b>Q05.a</b><br> <p>Rotation 120°   Modified tile</p>                                                                                                                                                                                           | <b>Q05.b</b><br> <p>Rotation 120°   Modified tile</p>                                                                                                                                                                                              |
| Fundamental region |  <p>Reflection</p>  <p>3-fold rotation</p>  <p>Fundamental region</p> |  <p>3 reflections</p>  <p>3-fold rotation</p>  <p>Fundamental region</p> |
| The tiling         |                                                                                                                                                            |                                                                                                                                                                                                                                                                                                                                         |

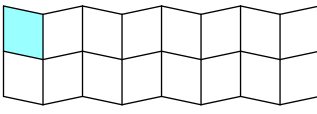
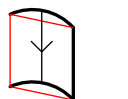
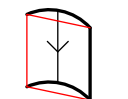
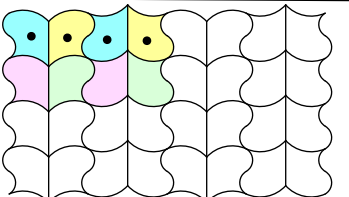
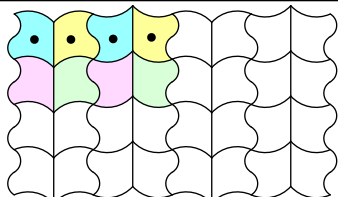
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| The prototype      | <b>Q05</b><br>Choices: 2 lengths<br>$Pn=2$<br>Operations: 4-fold rotation, 2 reflections<br>  |                                                                                                                                      |                              |                                                                                                  |
| Modification       | <b>Q06.a</b><br> Rotation 90°                                                                                                                                                  |  Modified tile                                      | <b>Q06.b</b><br> Rotation 90° |  Modified tile |
| Fundamental region |  Reflection   4-fold rotation   Fundamental region                                                                                                                             |  Reflection   4-fold rotation   Fundamental region |                                                                                                                |                                                                                                  |
| The tiling         |                                                                                                                                                                                |                                                                                                                                      |                              |                                                                                                  |

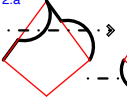
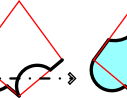
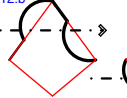
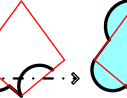
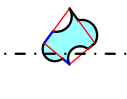
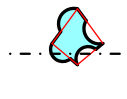
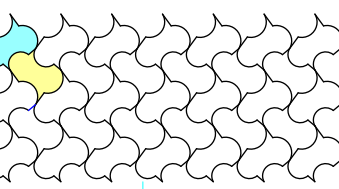
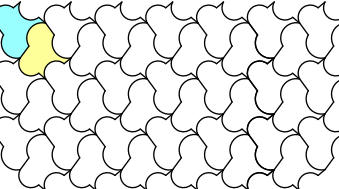
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| The prototype      | <b>Q06</b><br>Choices: 2 lengths<br>$Pn=2$<br>Operations: 2 glide reflections<br>  |                                                                                                                                             |                                   |                                                                                                   |
| Modification       | <b>Q07.a</b><br> Glide reflection                                                                                                                                     |  Modified tile                                           | <b>Q07.b</b><br> Glide reflection |  Modified tile |
| Fundamental region |  Glide reflection   2-fold rotation   Fundamental region                                                                                                              |  Glide reflection   2-fold rotation   Fundamental region |                                                                                                                      |                                                                                                   |
| The tiling         |                                                                                                                                                                       |                                                                                                                                             |                                   |                                                                                                   |

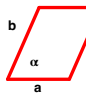
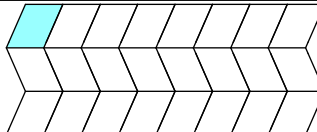
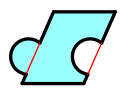
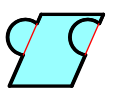
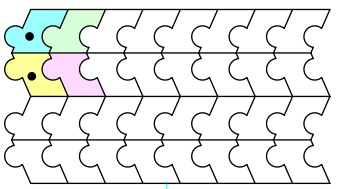
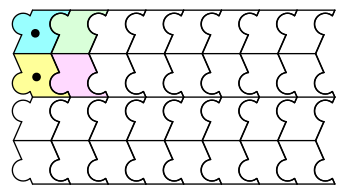
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| The prototype      | <b>Q07</b><br>Choices: 3 lengths<br>$Pn=3$<br>Operations: 3 reflections, 2-fold rotation<br>  |                                                                                                                                               |                               |                                                                                                    |
| Modification       | <b>Q08.a</b><br> Rotation 90°                                                                                                                                                  |  Modified tile                                             | <b>Q08.b</b><br> Rotation 90° |  Modified tile |
| Fundamental region |  2-fold rotation   Reflection   Reflection   Funda. region                                                                                                                     |  2-fold rotation   Reflection   Reflection   Fund. region |                                                                                                                  |                                                                                                    |
| The tiling         |                                                                                                                                                                                |                                                                                                                                               |                              |                                                                                                    |

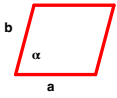
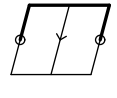
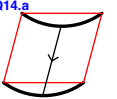
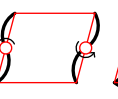
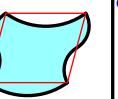
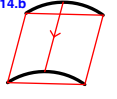
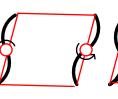
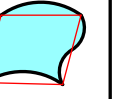
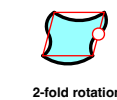
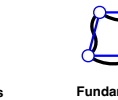
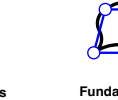
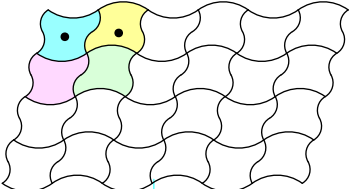
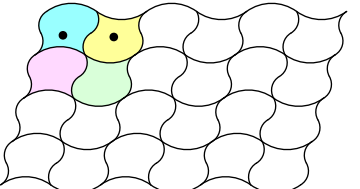
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| The prototype      | <b>Q08</b><br>Choices: 2 lengths, 1 angle, $Pn=3$<br>Operations: 2 translations<br>  |                                                                                                          |                               |                                                                                                     |
| Modification       | <b>Q09.a</b><br> Translation                                                                                                                                            |  Modified tile      | <b>Q09.b</b><br> Translation |  Modified tile |
| Fundamental region |  Fundamental region                                                                                                                                                     |  Fundamental region |                                                                                                                   |                                                                                                     |
| The tiling         |                                                                                                                                                                         |                                                                                                          |                              |                                                                                                     |

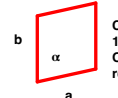
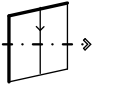
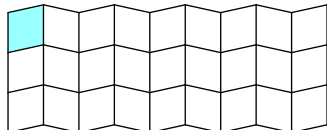
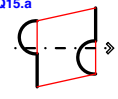
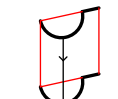
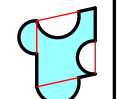
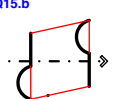
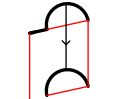
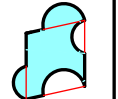
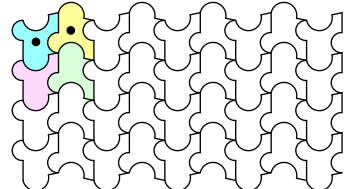
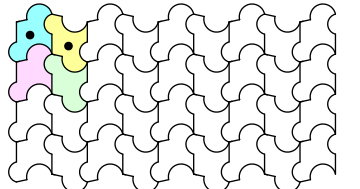
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| The prototype      | <b>Q09</b><br><br>Choise: 2 lengths, one angle, $Pn=3$<br>Operations: 1 glide, 2 reflections |                                                                                                    |                                                             |                                                                                                     |
| Modification       | <b>Q10.a</b><br><br>Glide reflection                                                         | <br>Modified tile | <b>Q10.b</b><br><br>Glide reflection                         | <br>Modified tile |
| Fundamental region | <br>Glide reflection      Reflection      Fundamental region                                 |                                                                                                    | <br>Glide reflection      Reflection      Fundamental region |                                                                                                     |
| The tiling         |                                                                                              |                                                                                                    |                                                             |                                                                                                     |

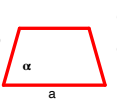
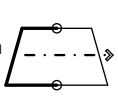
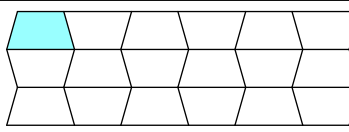
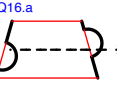
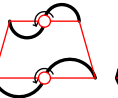
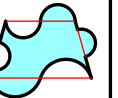
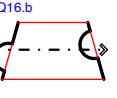
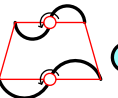
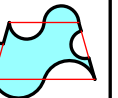
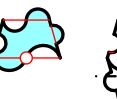
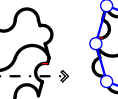
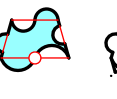
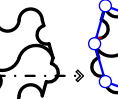
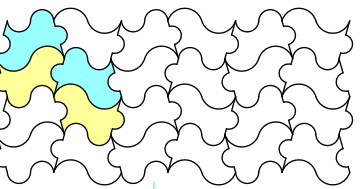
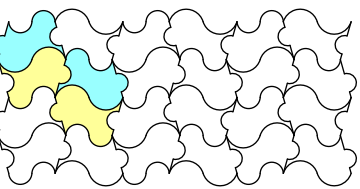
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| The prototype      | <b>Q10</b><br><br>Choices: 2 lengths and an angle $Pn=3$<br>Operations: 2-fold rotation, translation |                                                                                                    |                                                             |                                                                                                    |
| Modification       | <b>Q11.a</b><br><br>2-fold rotation                                                                  | <br>Translation | <b>Q11.b</b><br><br>2-fold rotation                         | <br>Translation |
| Fundamental region | <br>Reflection      2-fold rotation      Fundamental region                                          |                                                                                                    | <br>Reflection      2-fold rotation      Fundamental region |                                                                                                    |
| The tiling         |                                                                                                      |                                                                                                    |                                                             |                                                                                                    |

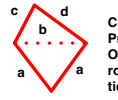
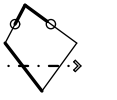
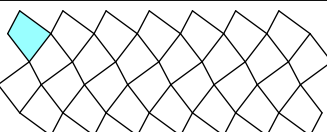
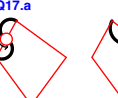
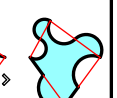
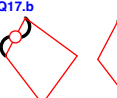
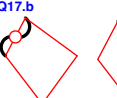
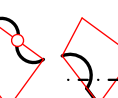
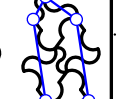
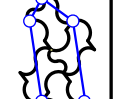
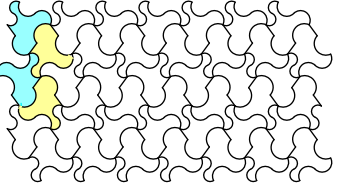
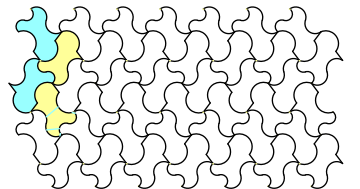
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| The prototype      | <b>Q11</b><br><br>Choises: 3 lengths, $Pn=3$<br>Operatos: 2 glide reflections |                                                                                                      |                                              |                                                                                                       |
| Modification       | <b>Q12.a</b><br><br>Glide reflection                                         | <br>Modified tile | <b>Q12.b</b><br><br>Glide reflection         | <br>Modified tile |
| Fundamental region | <br>Glide reflection      Fundamental region                                 |                                                                                                      | <br>Glide reflection      Fundamental region |                                                                                                       |
| The tiling         |                                                                              |                                                                                                      |                                             |                                                                                                       |

|                    |                                                                                                                                                                                    |                                                                                                        |                                                                                                                             |                                                                                                        |
|--------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|
| The prototype      | <b>Q12</b><br><br>Choises: 2 lengths, 1 angle $Pn=3$<br>Operations: 2 reflections, translation |                                                                                                        |                                         |                                                                                                        |
| Modification       | <b>Q13.a</b><br><br>Translation                                                               | <br>Modified tile | <b>Q13.b</b><br><br>Translation        | <br>Modified tile |
| Fundamental region | <br>Reflection      Fundamental region                                                        |                                                                                                        | <br>Reflection      Fundamental region |                                                                                                        |
| The tiling         |                                                                                               |                                                                                                        |                                        |                                                                                                        |

|                    |                                                                                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                                                                                                           |
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| The prototype      | <b>Q13</b><br> <p>Choices: 2 lengths<br/>1 angle <math>Pn=3</math><br/>Operations: 2 2-fold rotations, translation</p>   |                                                                                                                                                                                                                                                                                                                           |
| Modification       | <b>Q14.a</b><br> Translation<br> 2 2-fold rotations<br> Modified tile                                                     | <b>Q14.b</b><br> Translation<br> 2 2-fold rotations<br> Modified tile |
| Fundamental region |  2-fold rotations<br> Fundamental region                                                                                                                                                                   |  2-fold rotations<br> Fundamental region                                                                                                                |
| The tiling         |                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                           |

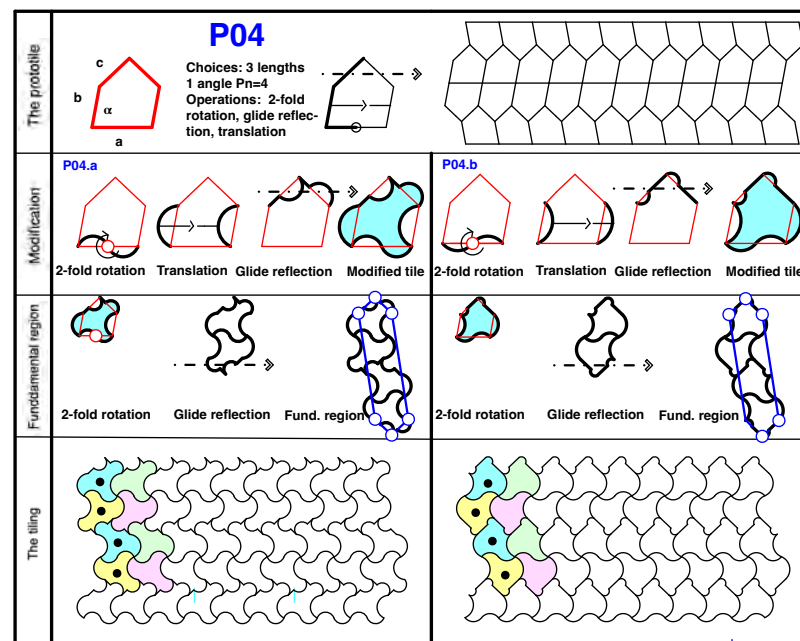
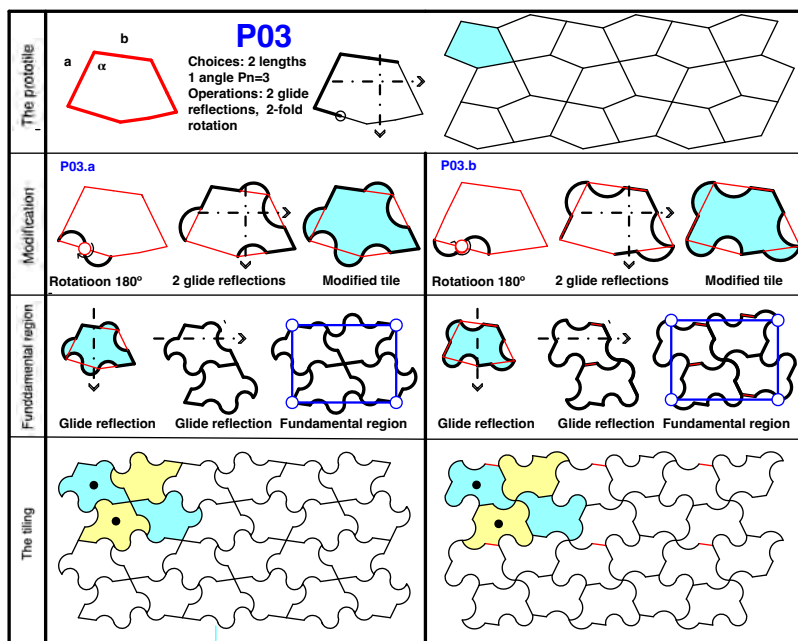
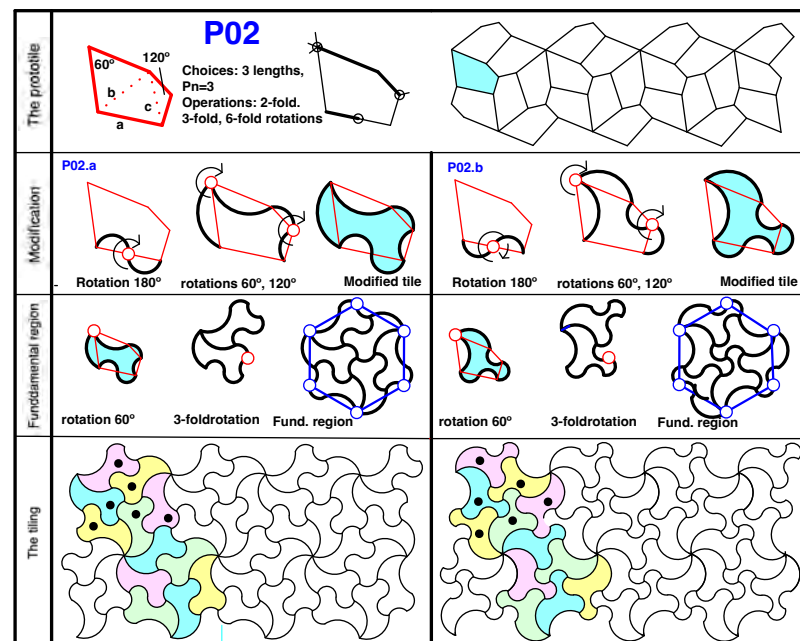
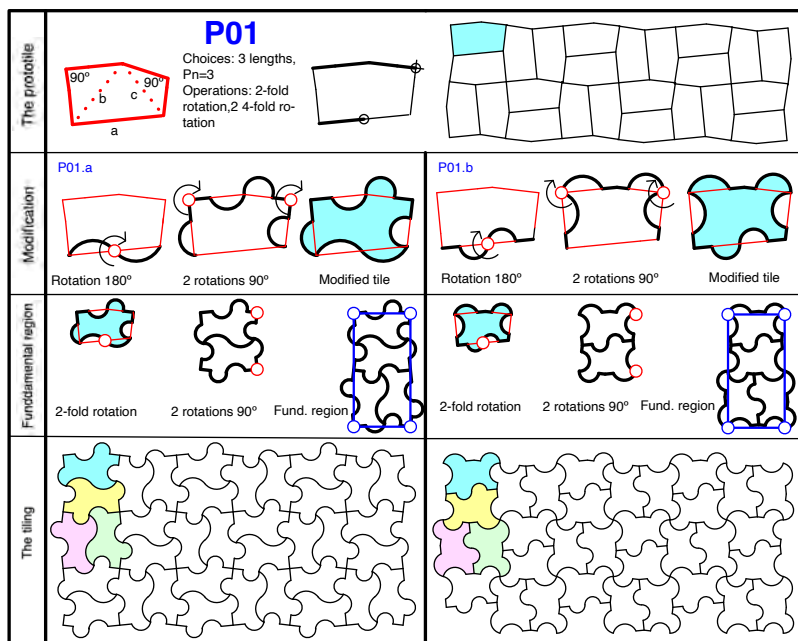
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|--------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| The prototype      | <b>Q14</b><br> <p>Choices: 2 lengths<br/>1 angle <math>Pn=3</math><br/>Operations: glide reflection, translation</p>   |                                                                                                                                                                                                                                                                                                                              |
| Modification       | <b>Q15.a</b><br> Glide reflection<br> Translation<br> Modified tile                                                    | <b>Q15.b</b><br> Glide reflection<br> Translation<br> Modified tile |
| Fundamental region |  Glide reflection<br> Fundamental region                                                                                                                                                                  |  Glide reflection<br> Fundamental region                                                                                                               |
| The tiling         |                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                              |

|                    |                                                                                                                                                                                                                                                                                                                                                                                      |                                                                                                                                                                                                                                                                                                                                      |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| The prototype      | <b>Q15</b><br> <p>Choices: 2 lengths<br/>1 angle <math>Pn=3</math><br/>Operations: 2 2-fold rotations, glide reflection</p>   |                                                                                                                                                                                                                                                                                                                                      |
| Modification       | <b>Q16.a</b><br> Glide reflection<br> 2 2-fold rotations<br> Modified tile                                                  | <b>Q16.b</b><br> Glide reflection<br> 2 2-fold rotations<br> Modified tile |
| Fundamental region |  2-fold rotation<br> Glide reflection<br> Fund. region                                                                      |  2-fold rotation<br> Glide reflection<br> Fund. region                     |
| The tiling         |                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                                                                                      |

|                    |                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                                                                                                                                                                                               |
|--------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| The prototype      | <b>Q16</b><br> <p>Choices: 4 lengths<br/><math>Pn=4</math><br/>Operations: 2 2-fold rotations, glide reflection</p>                                                               |                                                                                                                                                                                                                                                                                                                                                                                                                                               |
| Modification       | <b>Q17.a</b><br> 2-fold rotation<br> 2-fold rotation<br> Glide reflect.<br> Modified tile | <b>Q17.b</b><br> 2-fold rotation<br> 2-fold rotation<br> Glide reflect.<br> Modified tile |
| Fundamental region |  Glide reflect. 2-fold rotation<br> Glide reflect.<br> Fund. r.                                                                                                                |  Glide reflect. 2-fold rotation<br> Glide reflect.<br> Fund. r.                                                                                                                |
| The tiling         |                                                                                                                                                                                                                                                                     |                                                                                                                                                                                                                                                                                                                                                                                                                                               |

|                    |                  |                  |  |  |
|--------------------|------------------|------------------|--|--|
| The prototype      | <b>Q17</b><br>   |                  |  |  |
| Modification       | <b>Q18.a</b><br> | <b>Q18.b</b><br> |  |  |
| Fundamental region |                  |                  |  |  |
| The tiling         |                  |                  |  |  |

|                    |                  |                  |  |  |
|--------------------|------------------|------------------|--|--|
| The prototype      | <b>Q18</b><br>   |                  |  |  |
| Modification       | <b>Q19.a</b><br> | <b>Q19.b</b><br> |  |  |
| Fundamental region |                  |                  |  |  |
| The tiling         |                  |                  |  |  |



|                    |                                                                                                            |                      |                   |
|--------------------|------------------------------------------------------------------------------------------------------------|----------------------|-------------------|
| The prototype      | <b>P05</b><br>Choose: 3 lengths<br>1 angle $Pn=4$<br>Operations: glide reflection, translation, reflection |                      |                   |
| Modification       | <b>P05.a</b><br><br>Translation                                                                            | <br>Glide reflection | <br>Modified tile |
| Fundamental region | <br>Reflection                                                                                             | <br>Fund. region     |                   |
| The tiling         | <br>Translation                                                                                            |                      | <br>Translation   |

|                    |                                                                                                  |                        |                   |
|--------------------|--------------------------------------------------------------------------------------------------|------------------------|-------------------|
| The prototype      | <b>P06</b><br>Choose: 4 lengths<br>1 angle $Pn=5$<br>Operations: translation, 3 2-fold rotations |                        |                   |
| Modification       | <b>P06.a</b><br><br>Translation                                                                  | <br>3 2-fold rotations | <br>Modified tile |
| Fundamental region | <br>2-fold rotation                                                                              | <br>Fund. region       |                   |
| The tiling         | <br>Translation                                                                                  |                        | <br>Translation   |

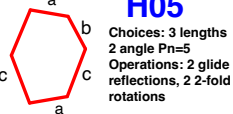
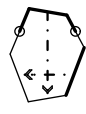
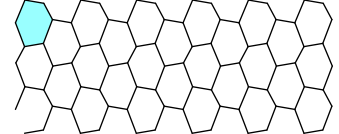
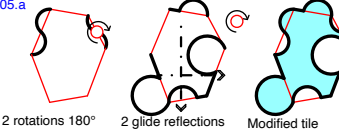
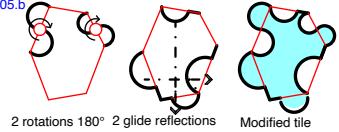
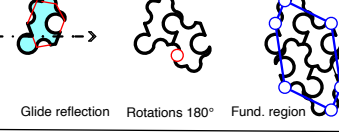
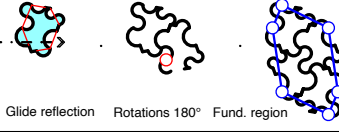
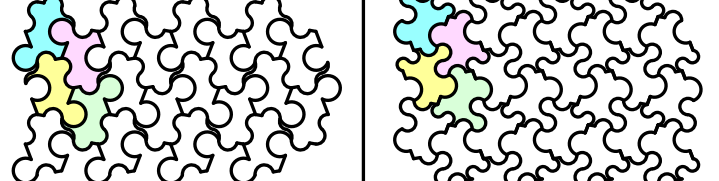
|                    |                                                                                                               |                        |                   |
|--------------------|---------------------------------------------------------------------------------------------------------------|------------------------|-------------------|
| The prototype      | <b>P07</b><br>Choose: 4 lengths<br>1 angle $Pn=45$<br>Operations: translation, 2 2-fold rotations, reflection |                        |                   |
| Modification       | <b>P07.a</b><br><br>Translation                                                                               | <br>2 2-fold rotations | <br>Modified tile |
| Fundamental region | <br>Reflection                                                                                                | <br>2-fold rotation    | <br>Fund. region  |
| The tiling         | <br>Translation                                                                                               |                        | <br>Translation   |

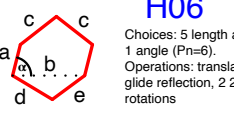
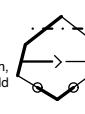
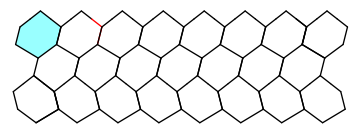
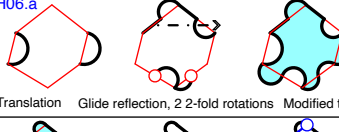
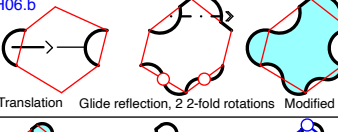
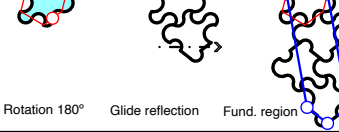
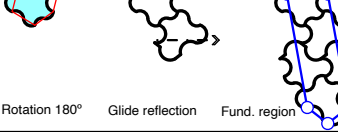
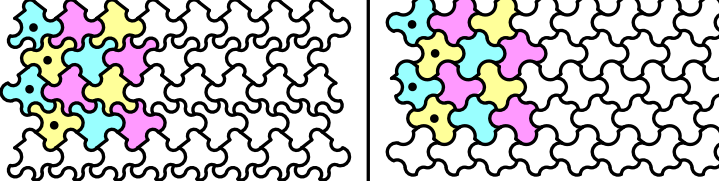
|                    |                                                                                                    |                           |                                         |                           |
|--------------------|----------------------------------------------------------------------------------------------------|---------------------------|-----------------------------------------|---------------------------|
| The prototile      | <b>H01</b><br><p>Choices: 3 lengths<br/> <math>Pn=3</math><br/> Operations: 3 3-fold rotations</p> |                           |                                         |                           |
| Modification       | <b>H01.a</b><br><p>3 rotations 180°</p>                                                            | <p>Modified tile</p>      | <b>H01.b</b><br><p>3 rotations 180°</p> | <p>Modified tile</p>      |
| Fundamental region | <p>3-fold rotation</p>                                                                             | <p>Fundamental region</p> | <p>3-fold rotation</p>                  | <p>Fundamental region</p> |
| The tiling         |                                                                                                    |                           |                                         |                           |

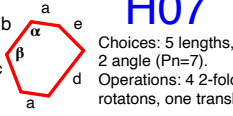
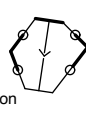
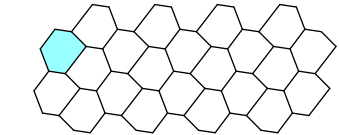
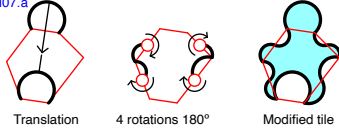
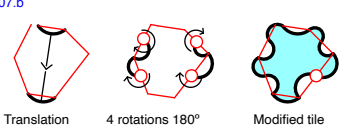
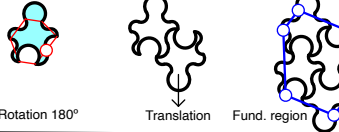
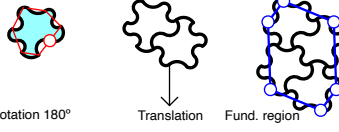
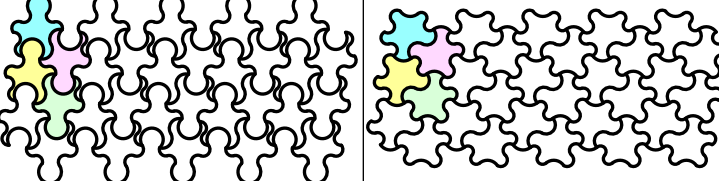
|                    |                                                                                                                            |                            |                         |                                    |                            |                      |
|--------------------|----------------------------------------------------------------------------------------------------------------------------|----------------------------|-------------------------|------------------------------------|----------------------------|----------------------|
| The prototile      | <b>H02</b><br><p>Choices: 3 lengths and 2 angles (<math>Pn=5</math>)<br/> Operations: 2 glide reflections, translation</p> |                            |                         |                                    |                            |                      |
| Modification       | <b>H02.a</b><br><p>Translation</p>                                                                                         | <p>2 glide reflections</p> | <p>Modified tile</p>    | <b>H02.b</b><br><p>Translation</p> | <p>2 glide reflections</p> | <p>Modified tile</p> |
| Fundamental region | <p>Glide reflection</p>                                                                                                    | <p>Fundamental region</p>  | <p>Glide reflection</p> | <p>Fundamental region</p>          |                            |                      |
| The tiling         |                                                                                                                            |                            |                         |                                    |                            |                      |

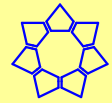
|                    |                                                                                                                            |                            |                         |                                    |                            |                      |
|--------------------|----------------------------------------------------------------------------------------------------------------------------|----------------------------|-------------------------|------------------------------------|----------------------------|----------------------|
| The prototile      | <b>H03</b><br><p>Choices: 3 lengths and 2 angles (<math>Pn=5</math>)<br/> Operations: 2 glide reflections, translation</p> |                            |                         |                                    |                            |                      |
| Modification       | <b>H03.a</b><br><p>Translation</p>                                                                                         | <p>2 glide reflections</p> | <p>Modified tile</p>    | <b>H03.b</b><br><p>Translation</p> | <p>2 glide reflections</p> | <p>Modified tile</p> |
| Fundamental region | <p>Glide reflection</p>                                                                                                    | <p>Fundamental region</p>  | <p>Glide reflection</p> | <p>Fundamental region</p>          |                            |                      |
| The tiling         |                                                                                                                            |                            |                         |                                    |                            |                      |

|                    |                                                                                           |  |                           |                                       |  |  |
|--------------------|-------------------------------------------------------------------------------------------|--|---------------------------|---------------------------------------|--|--|
| The prototile      | <b>H04</b><br><p>Choices: 3 lengths, 2 angles (<math>Pn=5</math>)<br/> 3 translations</p> |  |                           |                                       |  |  |
| Modification       | <b>H04.a</b><br><p>3 translations</p>                                                     |  |                           | <b>H04.b</b><br><p>3 translations</p> |  |  |
| Fundamental region | <p>Fundamental region</p>                                                                 |  | <p>Fundamental region</p> |                                       |  |  |
| The tiling         |                                                                                           |  |                           |                                       |  |  |

|                    |                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                    |
|--------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| The prototype      | <p><b>H05</b></p>  <p>Choices: 3 lengths<br/>2 angle <math>Pn=5</math><br/>Operations: 2 glide reflections, 2 2-fold rotations</p>   |                                                                                                                                                                                    |
| Modification       | <p><b>H05.a</b></p>  <p>2 rotations <math>180^\circ</math> 2 glide reflections Modified tile</p>                                                                                                                                                                                                        | <p><b>H05.b</b></p>  <p>2 rotations <math>180^\circ</math> 2 glide reflections Modified tile</p> |
| Fundamental region |  <p>Glide reflection Rotations <math>180^\circ</math> Fund. region</p>                                                                                                                                                                                                                                  |  <p>Glide reflection Rotations <math>180^\circ</math> Fund. region</p>                           |
| The tiling         |                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                    |

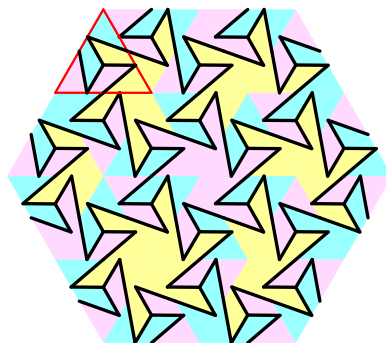
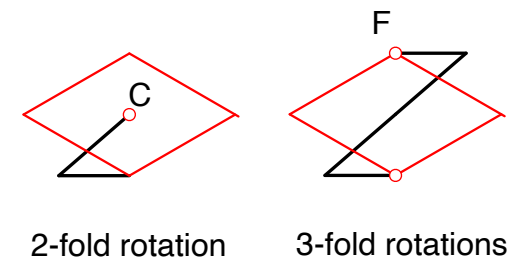
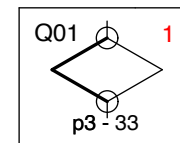
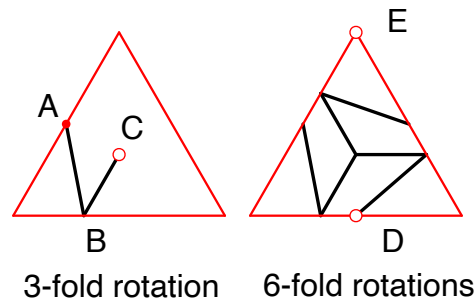
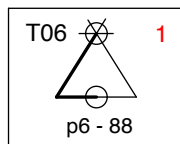
|                    |                                                                                                                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                               |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| The prototype      | <p><b>H06</b></p>  <p>Choices: 5 length and 1 angle <math>Pn=6</math><br/>Operations: translation, glide reflection, 2 2-fold rotations</p>   |                                                                                                                                                                               |
| Modification       | <p><b>H06.a</b></p>  <p>Translation Glide reflection, 2 2-fold rotations Modified tile</p>                                                                                                                                                                                                                          | <p><b>H06.b</b></p>  <p>Translation Glide reflection, 2 2-fold rotations Modified tile</p> |
| Fundamental region |  <p>Rotation <math>180^\circ</math> Glide reflection Fund. region</p>                                                                                                                                                                                                                                               |  <p>Rotation <math>180^\circ</math> Glide reflection Fund. region</p>                      |
| The tiling         |                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                                               |

|                    |                                                                                                                                                                                                                                                                                                                                                                                      |                                                                                                                                                                              |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| The prototype      | <p><b>H07</b></p>  <p>Choices: 5 lengths, 2 angle <math>Pn=7</math><br/>Operations: 4 2-fold rotations, one translation</p>   |                                                                                                                                                                              |
| Edge modifications | <p><b>H07.a</b></p>  <p>Translation 4 rotations <math>180^\circ</math> Modified tile</p>                                                                                                                                                                                                          | <p><b>H07.b</b></p>  <p>Translation 4 rotations <math>180^\circ</math> Modified tile</p> |
| Fundamental region |  <p>Rotation <math>180^\circ</math> Translation Fund. region</p>                                                                                                                                                                                                                                  |  <p>Rotation <math>180^\circ</math> Translation Fund. region</p>                         |
| The tiling         |                                                                                                                                                                                                                                                                                                  |                                                                                                                                                                              |

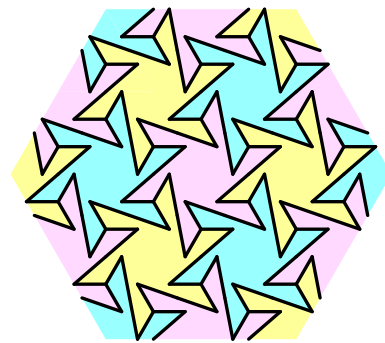


## 17. Two tiles, one tiling

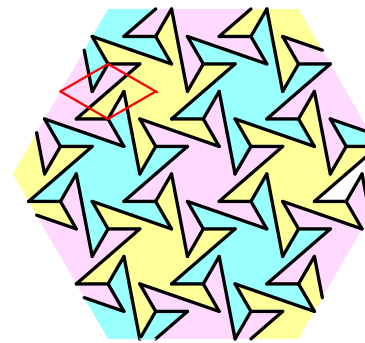
Here, we show how the same tiling may be created by starting with two different tiles, T06 and Q01. Begin with a motif of two connected segments, AB and BC. Point C is the center of the regular triangle, B is chosen freely, and point A is the midpoint of the side. Rotate ABC 3-fold around C, followed by either 2-fold rotations around points D or 6-fold around points E.



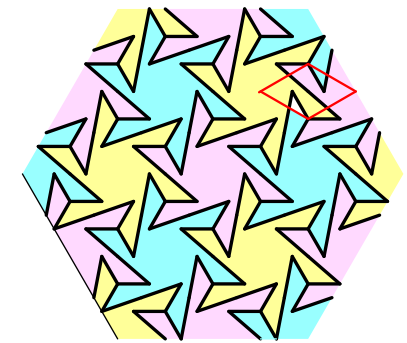
symmetrical coloration



"map" coloration

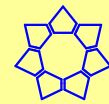


left version



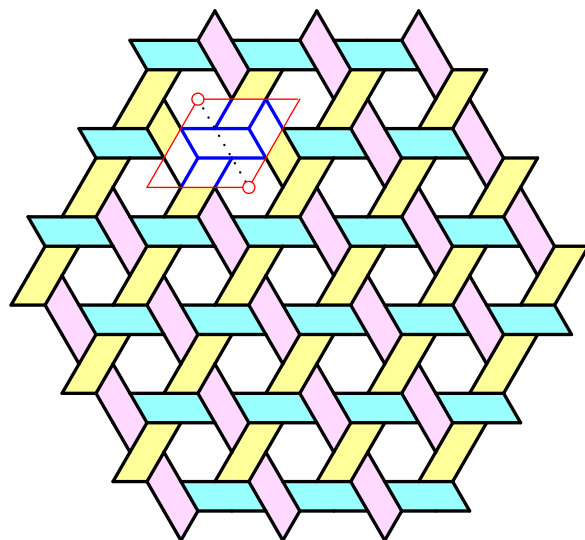
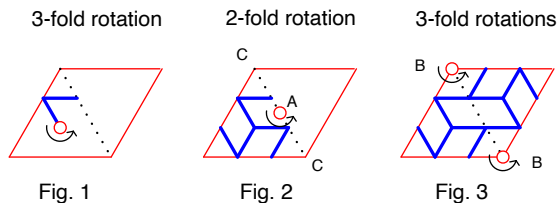
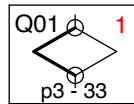
right version

We also illustrate that colouring the unit cell will not produce a correct "map" colouring. • The same motif is installed in a different orientation in the tile Q01. First, we rotate the motif 2-fold around the center C, followed by 3-fold rotations around the points F. • The tiling exists in two versions: since it is generated only by rotations (no reflections), they are the mirror images of each other or enantiomorphs.



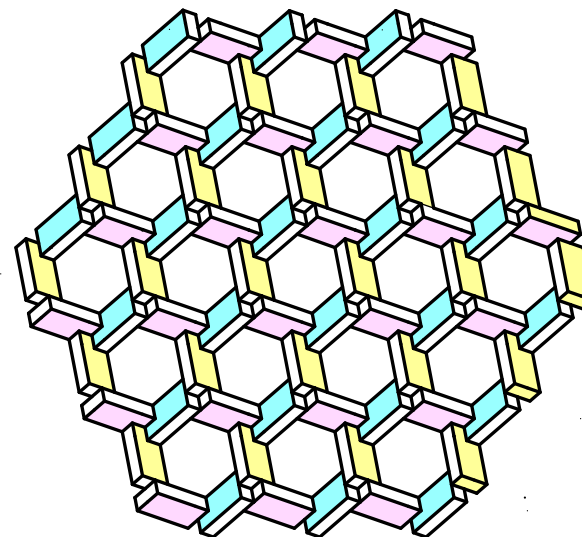
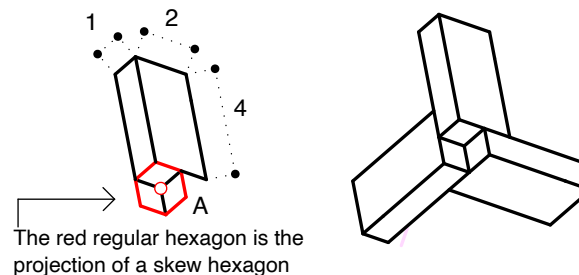
## 18. The pattern

Here, we present an example of viewing a 2-dimensional drawing of a pattern as a projection of a 3-D model. Create the blue motif inside the half-tile Q01 (Fig.1) and rotate it 3-fold around the center of the triangle (Fig. 2). The new motif is now 2-fold rotated around the center A, and we obtain the final motif in Fig. 3. Rotate 3-fold the final motif around the center B, and the pattern is obtained.



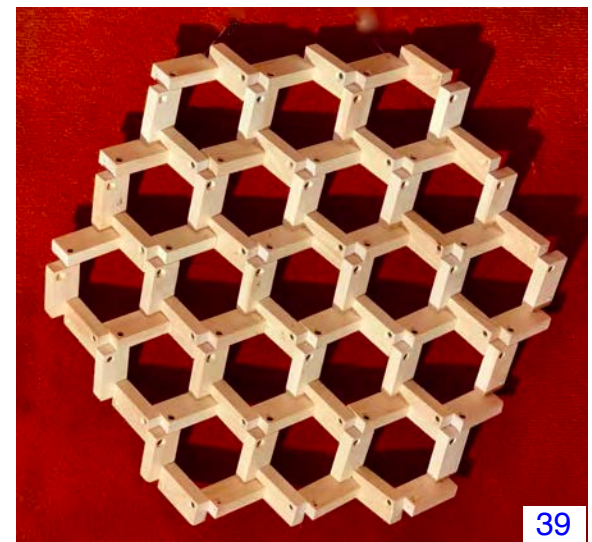
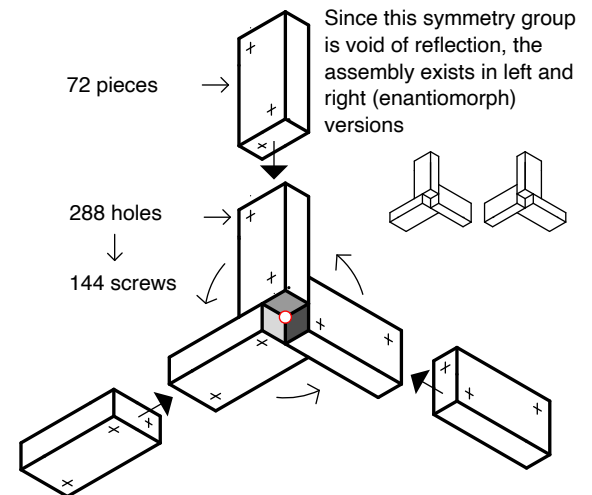
## the projection

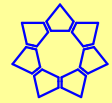
Any 2-D drawing may be viewed as a central, parallel or orthogonal projection of a 3-D configuration and can be lifted into 3-D. Looking at the pattern on the left, the colouring helps to recognize how the parallelograms (or the rectangles) are joined together in 3-D. To build a solid model, we added thickness to the plane figures. It was simple to prepare the orthogonal projection: the three identical rectangular prisms meet in a new type of joint at the common point A.



## the 3-D model

This joint is the product of a 3-D symmetry operation: a rectangular prism is rotated 3-fold around the “vertical” body-diagonal of the virtual cube defined by the three shaded faces.





## 19. Symmetrical tiles

PI 21

The 43 non-symmetrical tiles were presented on [PI 7](#); the 38 symmetrical tiles are shown on [PI 22](#). We separated the two groups because they offer distinct visual options in graphic design. Symmetrical tiles are recognizable: they contain either an axis of reflection separating two identical halves or a rotational center inside the tile.

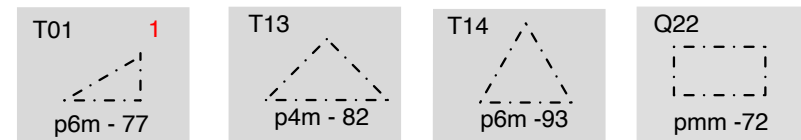
We opted for a simplified display of steps (see below) to construct the tiling on [PI 22](#) and also used a different system for modifying the edges. Instead of circular arcs, we used connected straight-line segments.

## 20. Mirror tiles

PI 22, PI 23

Tilings are generated with one or a combination of the four symmetry operations.

One asymmetrical tile from tile from [PI 7](#) and three symmetrical tiles from [PI 26](#) are grayed out because they tile the plane with reflection only. By adding a motif to these tiles, a wealth of patterns may be constructed digitally in a very efficient way



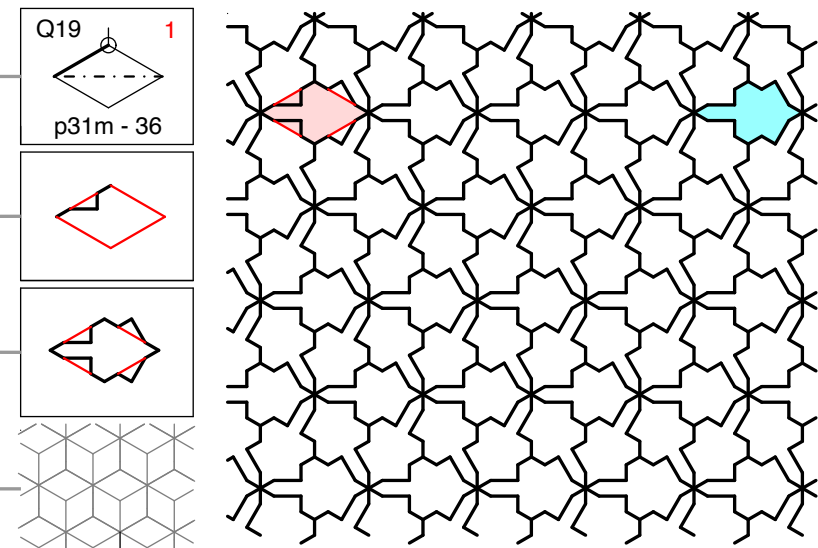
Example: -----

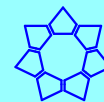
The geometric/structural model of the tile. The information includes our code (Q19), the parametric number (1), the symmetry group of the tiling (p31m) and the numbering by Grünbaum/Sheppard (36)

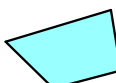
One edge is modified (replaced) by a polygonal line. For this group of tiles, the edges are replaced by connected segments. Again, for clarity, we opted for simplicity.

The modified edge is multiplied: rotated 120° (3-fold rotation), and the two copies are reflected about a x-x axis.

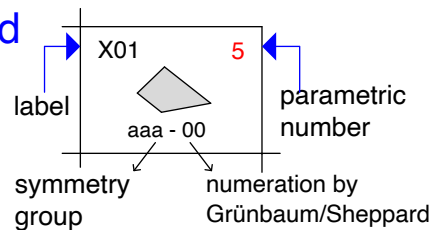
The lattice of the tiling; rhombi (120°-60) filling the plan.

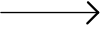
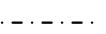
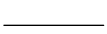
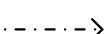
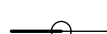




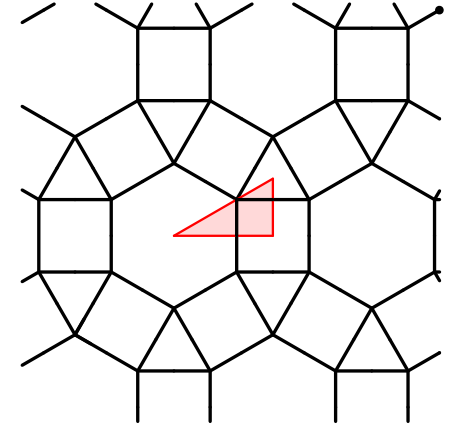
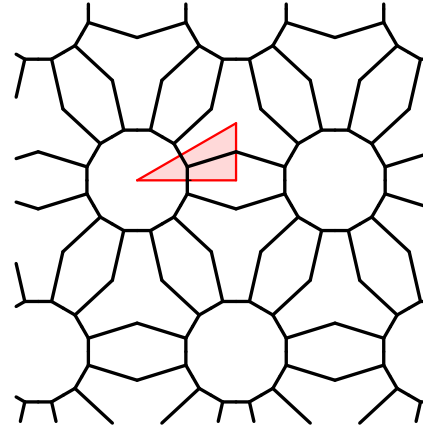
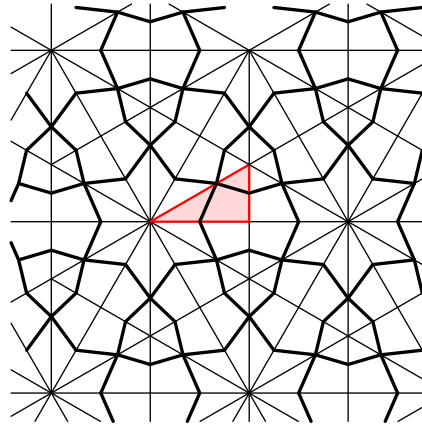
|                                                                                                            |                 |                 |                  |                 |                  |                 |                 |                 |                 |
|------------------------------------------------------------------------------------------------------------|-----------------|-----------------|------------------|-----------------|------------------|-----------------|-----------------|-----------------|-----------------|
| <br><b>Triangles</b>      | T12<br>p6m - 40 | T13<br>p4m - 82 | T14<br>p6m - 93  | T15<br>p6 - 90  | T16<br>cmm - 91  |                 |                 |                 |                 |
|                                                                                                            | Q19<br>p6m - 32 | Q20<br>p6m - 37 | Q21<br>p4m - 76  | Q22<br>pmm - 72 | Q23<br>p31m - 36 | Q24<br>p4 - 61  | Q25<br>p4 - 62  | Q26<br>p4g - 71 | Q27<br>p4g - 73 |
| <br><b>Quadrilaterals</b> | Q28<br>p6 - 34  | Q29<br>cm - 68  | Q30<br>pm - 64   | Q31<br>cmm - 74 | Q32<br>pmg - 66  | Q33<br>pgg - 59 | Q34<br>p2 - 57  | Q35<br>cmm - 67 | Q36<br>pmg - 69 |
|                                                                                                            | Q37<br>pmg - 58 |                 |                  |                 |                  |                 |                 |                 |                 |
| <br><b>Pentagons</b>      | P08<br>p4g - 29 | P09<br>cmm - 26 |                  |                 |                  |                 |                 |                 |                 |
| <br><b>Hexagons</b>     | H08<br>p6m - 20 | H09<br>p3 - 10  | H10<br>p31m - 18 | H11<br>p6 - 11  | H12<br>p31m - 16 | H13<br>cm - 14  | H014<br>cm - 12 | H15<br>cmm - 17 | H16<br>pmg - 15 |
|                                                                                                            | H17<br>pmg - 13 | H18<br>pgg - 9  | H19<br>p2 - 8    |                 |                  |                 |                 |                 |                 |

## Legend

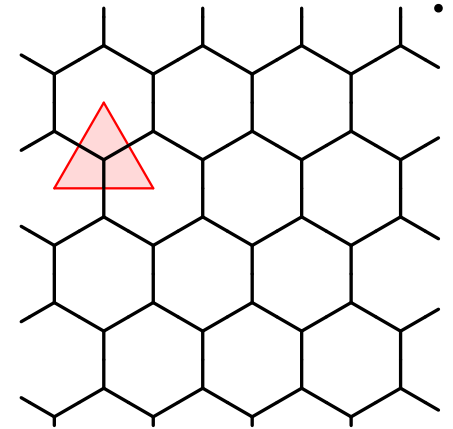
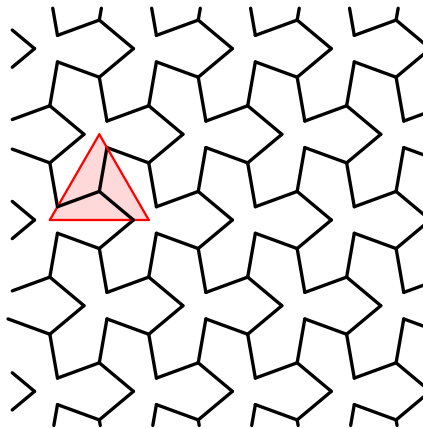
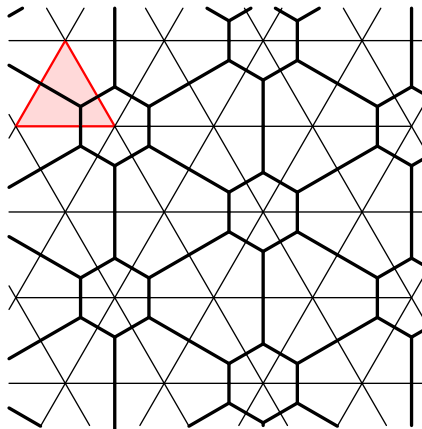


|            |        |                                                                                       |      |                    |                                                                                       |                      |                                                                                       |
|------------|--------|---------------------------------------------------------------------------------------|------|--------------------|---------------------------------------------------------------------------------------|----------------------|---------------------------------------------------------------------------------------|
| rotations: | 2-fold |  | 180° | translation        |  | modifiable edge      |  |
|            | 3-fold |  | 120° | reflection         |  | derived edge         |  |
|            | 4-fold |  | 90°  | transl. + reflect. |  | modifiable half-edge |  |
|            | 6-fold |  | 60°  | glided reflexion   |  | modifiable half-edge |  |

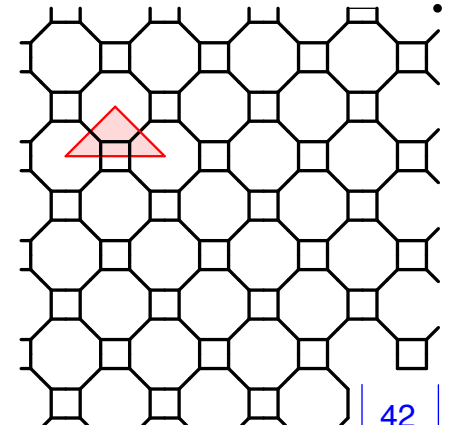
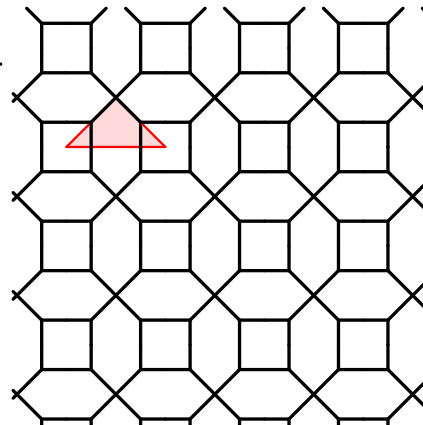
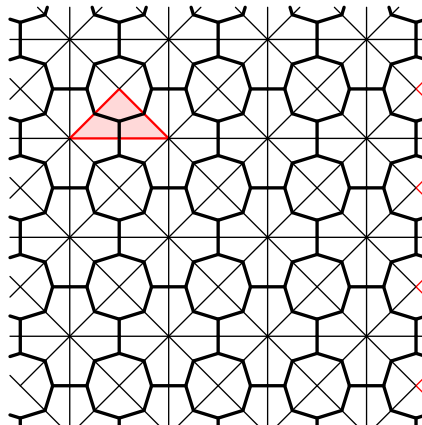
Group: p6m



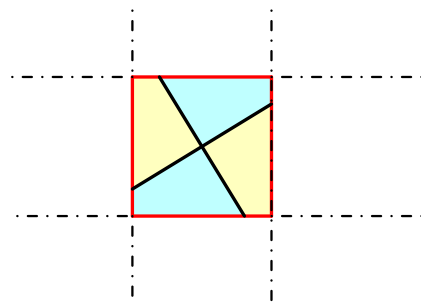
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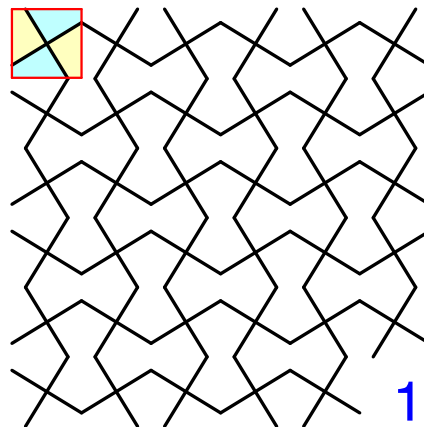
Group: p4m



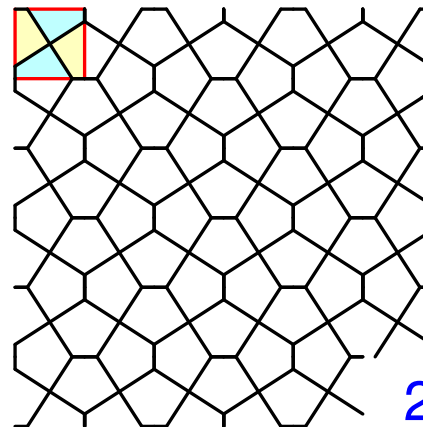
Group: pmm



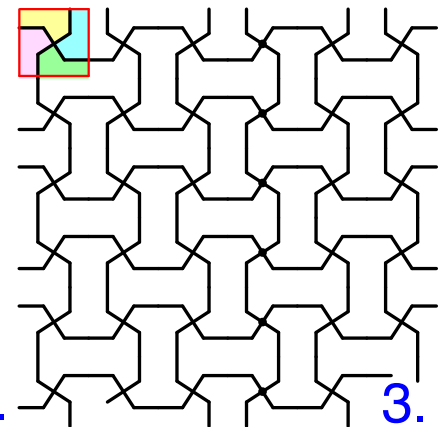
4 mirrors



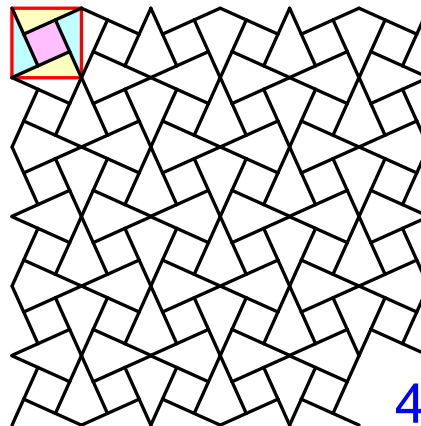
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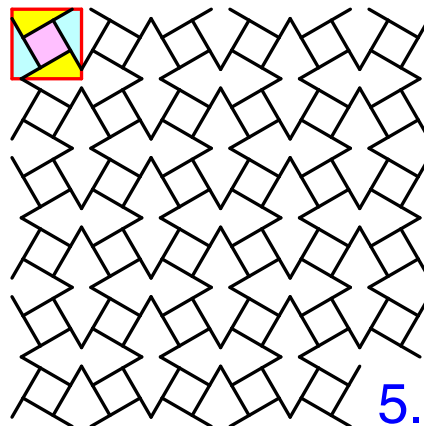
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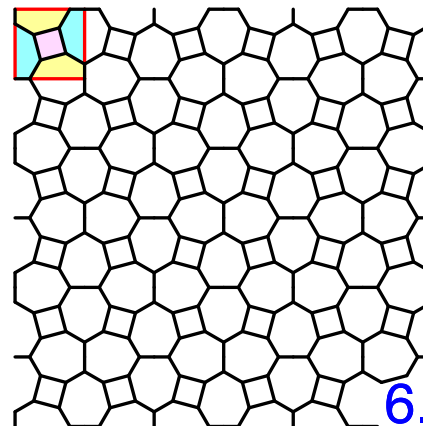
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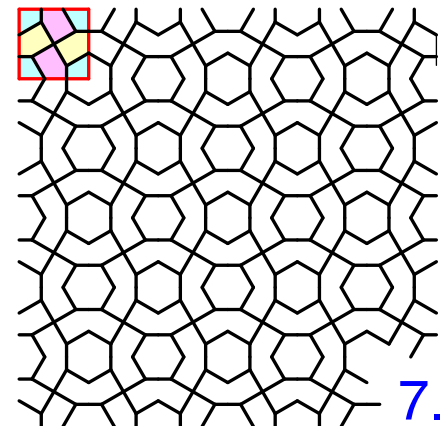
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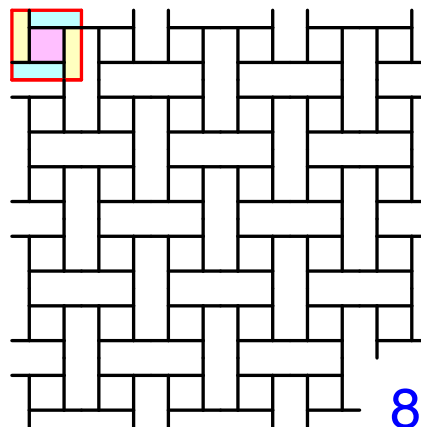
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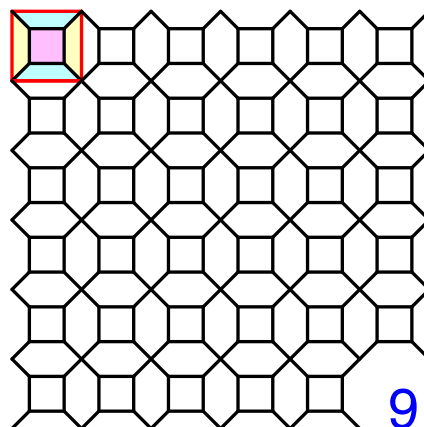
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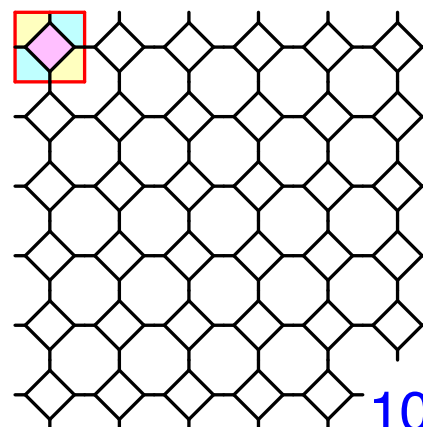
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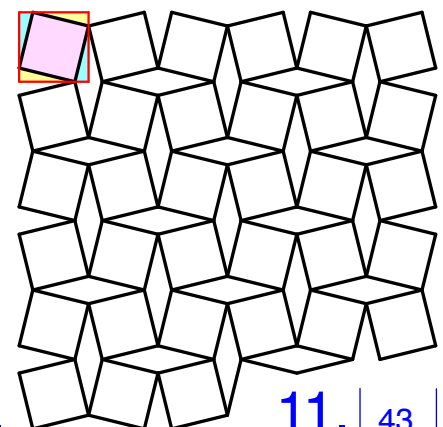
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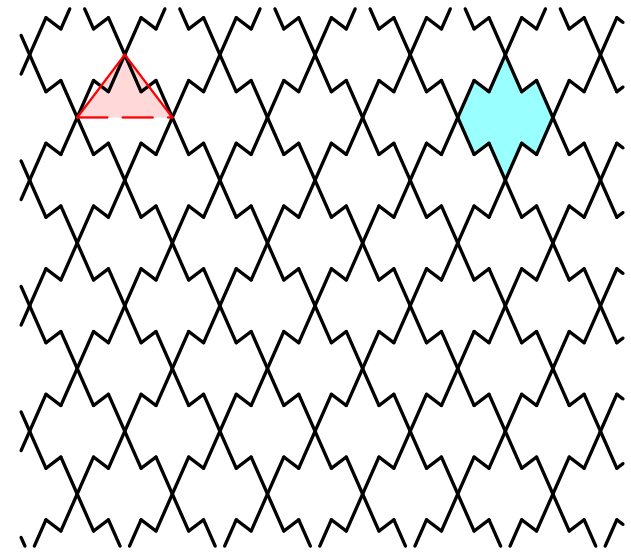
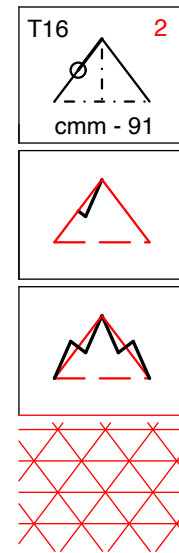
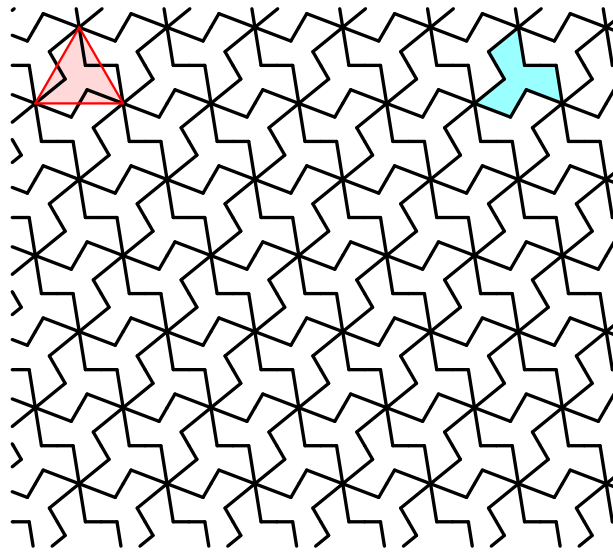
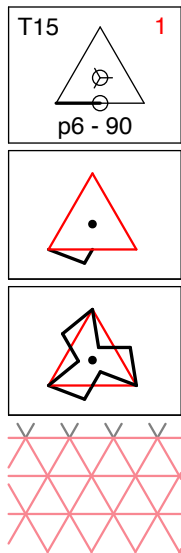
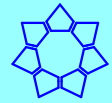
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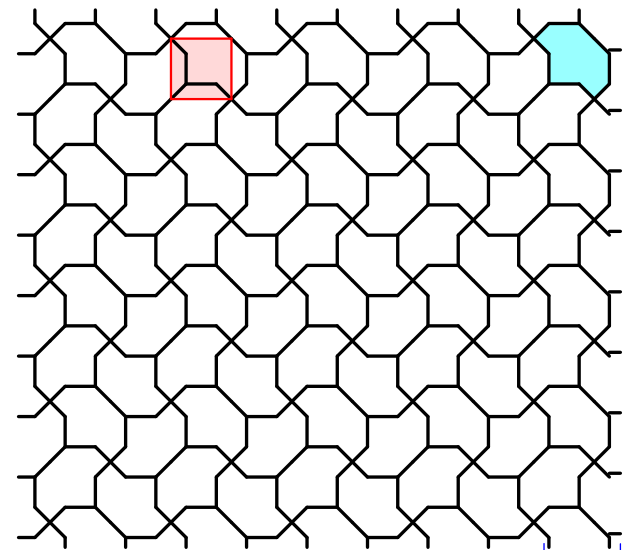
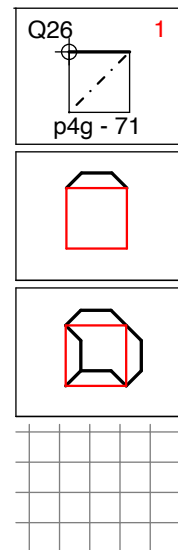
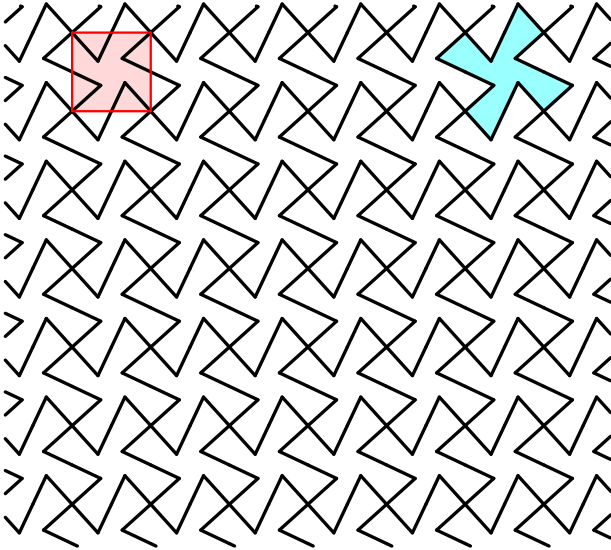
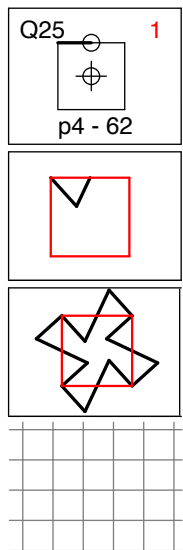
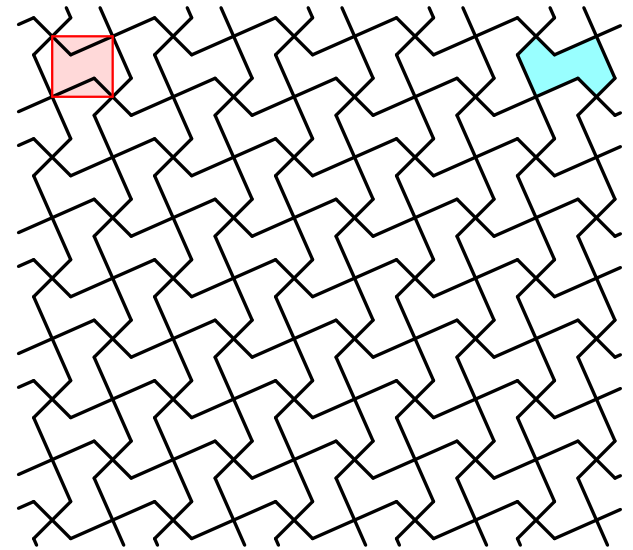
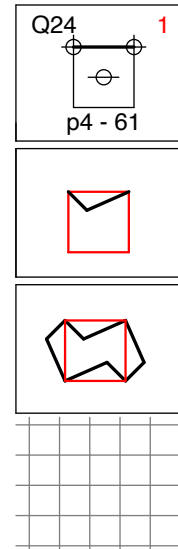
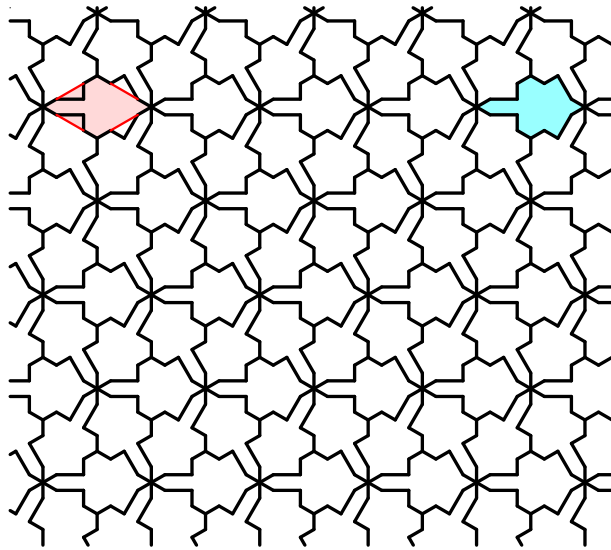
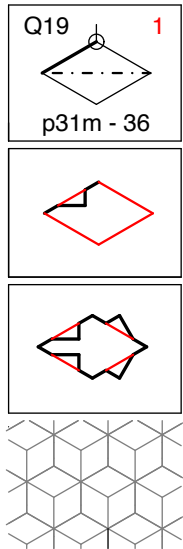
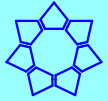


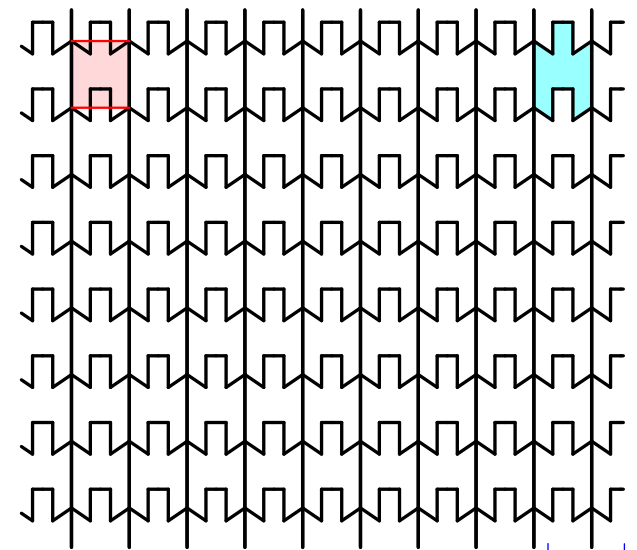
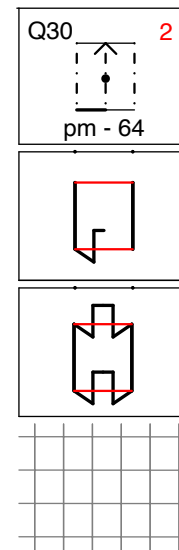
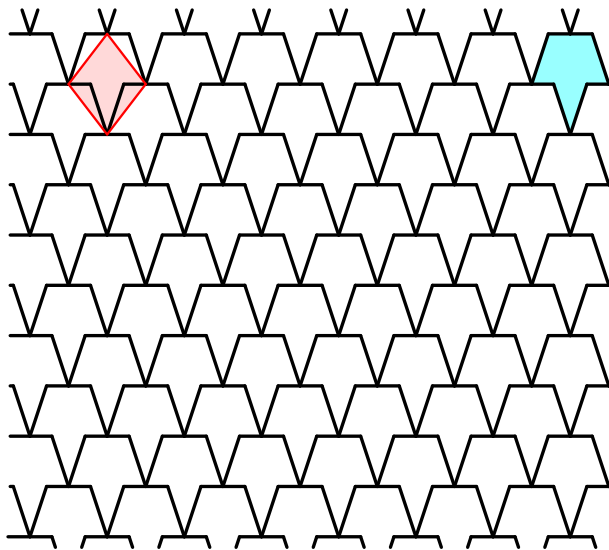
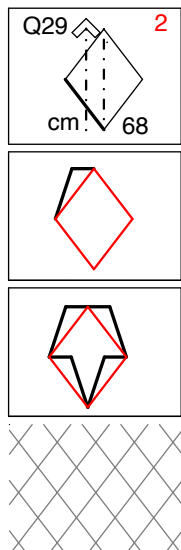
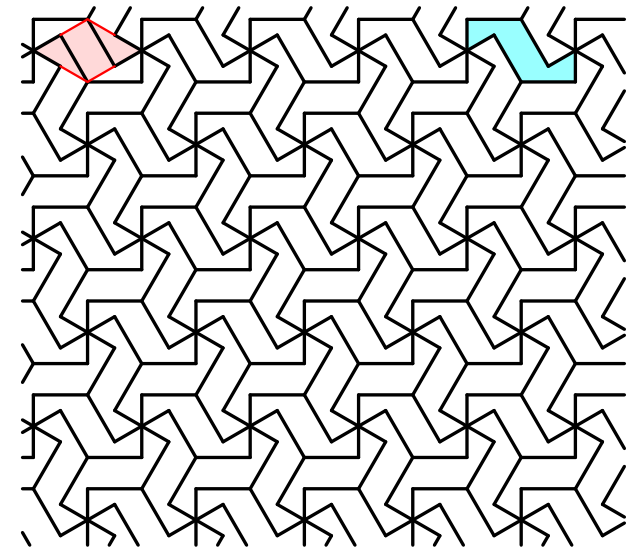
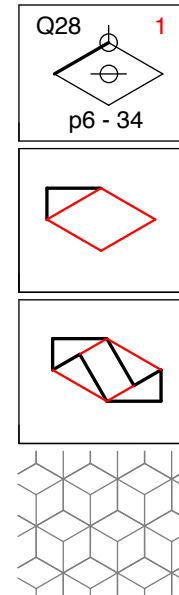
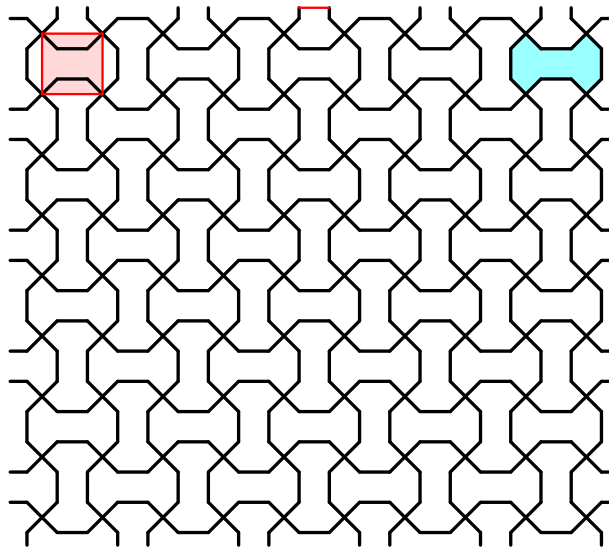
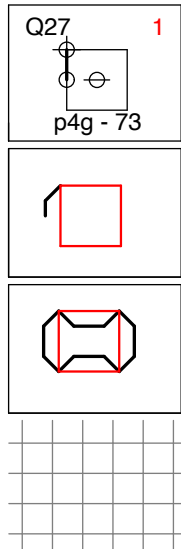
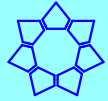
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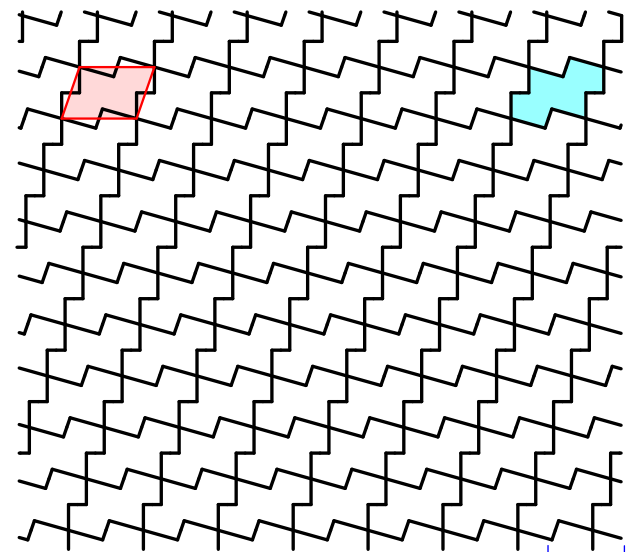
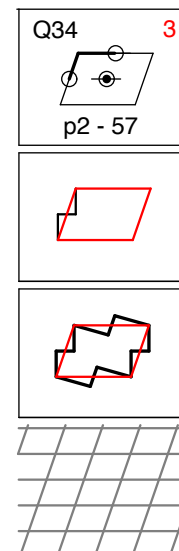
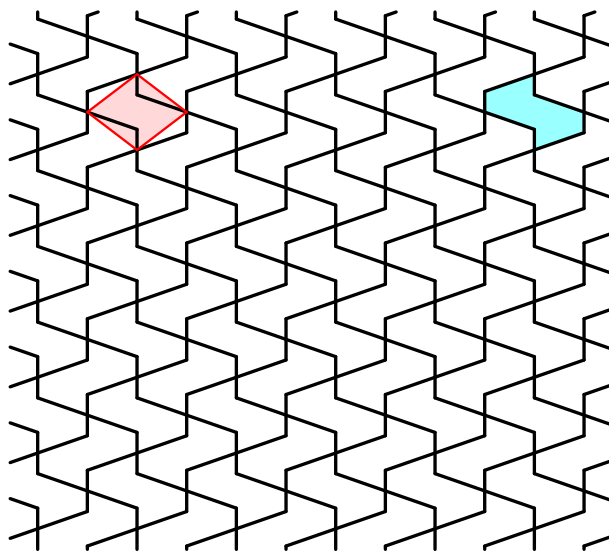
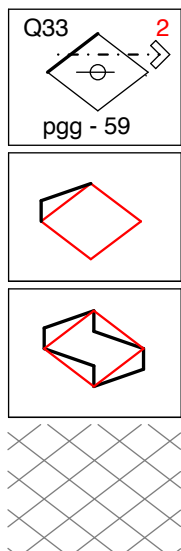
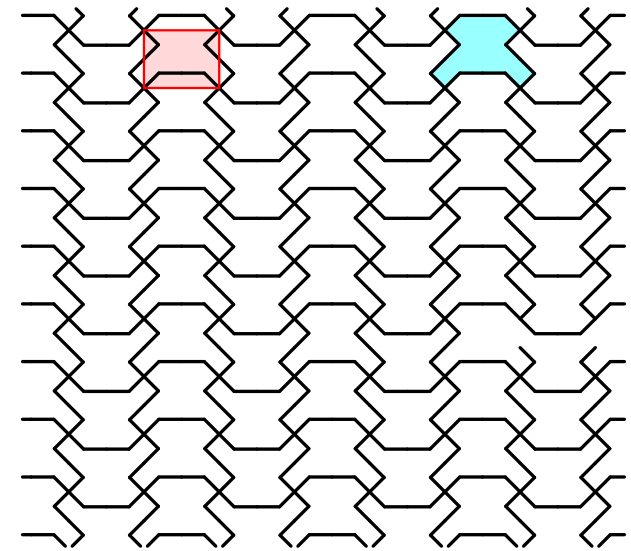
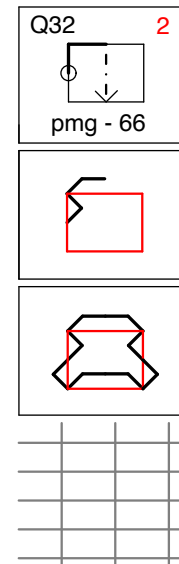
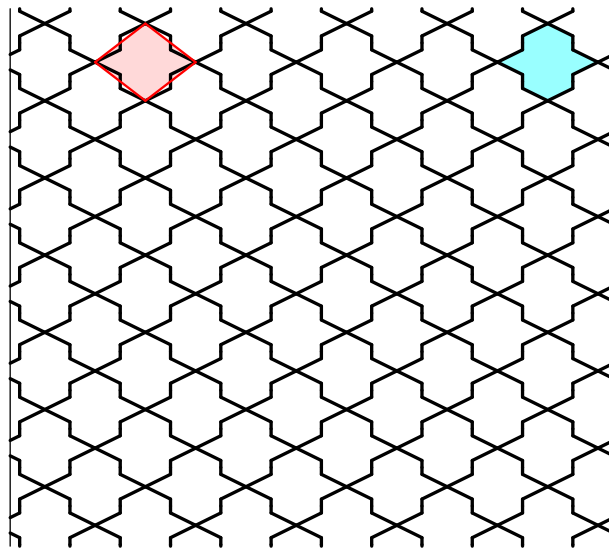
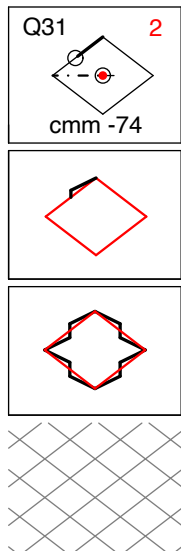
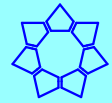


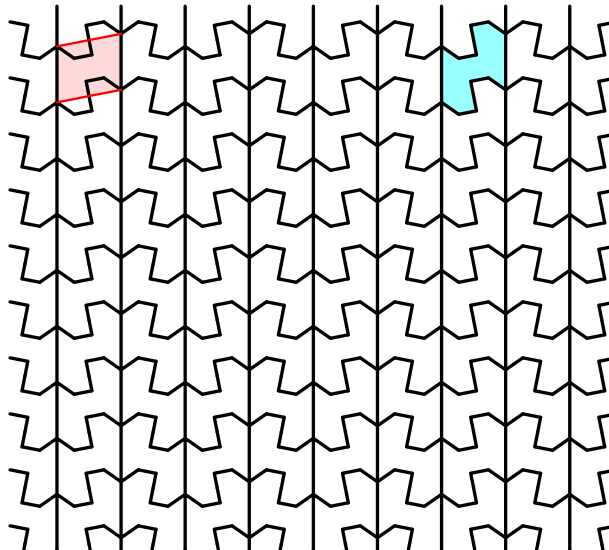
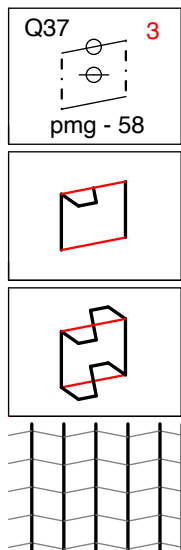
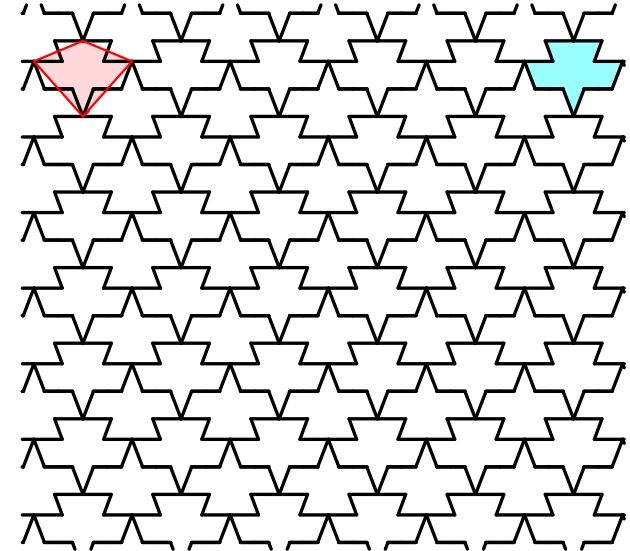
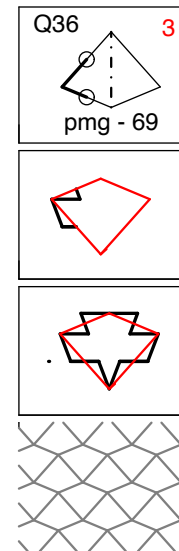
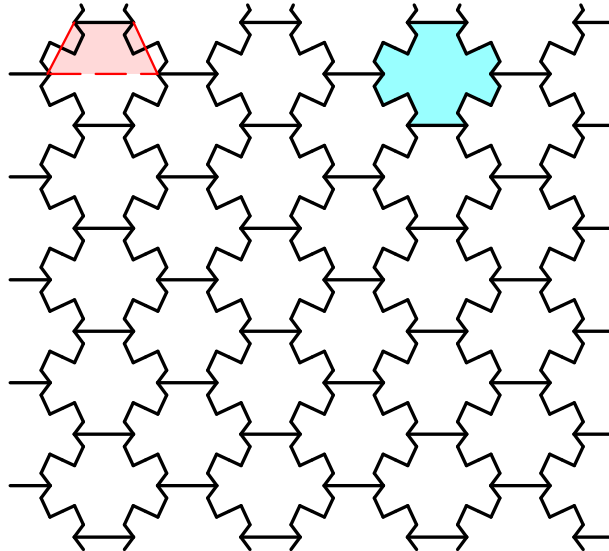
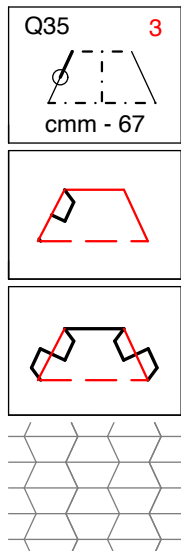
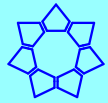
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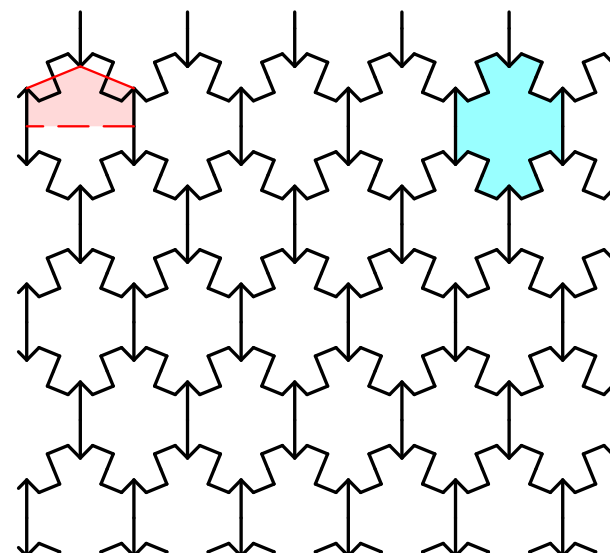
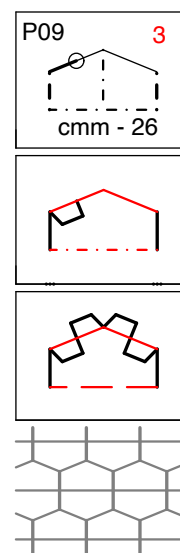
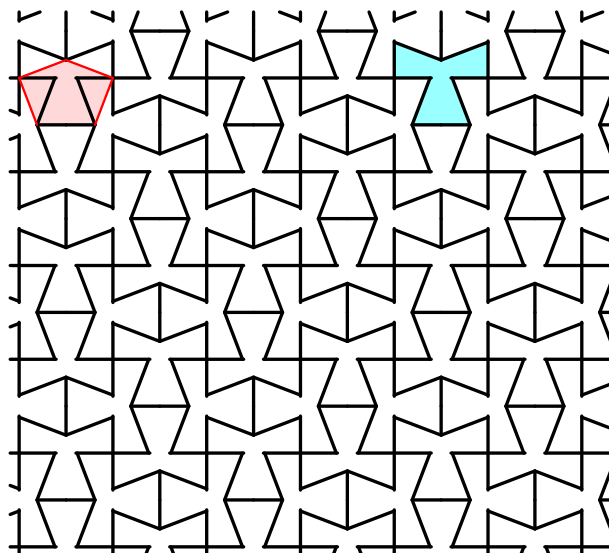
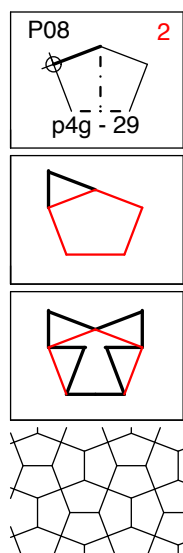
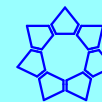


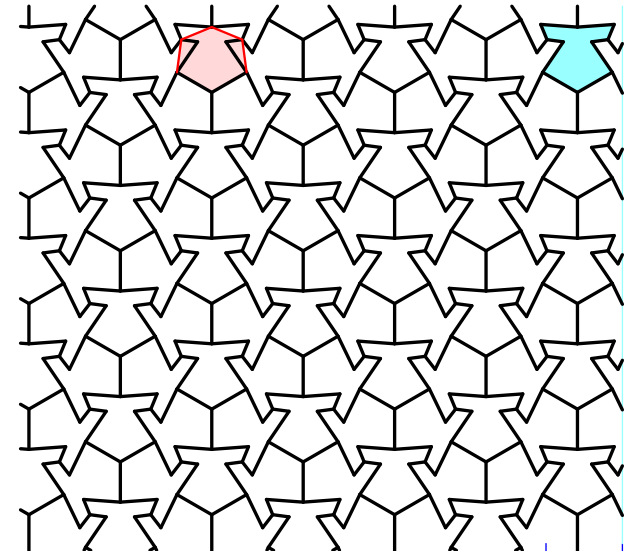
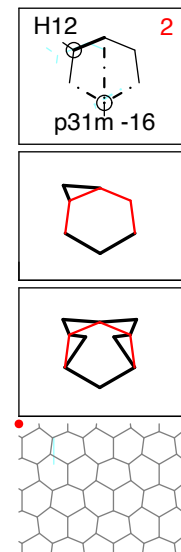
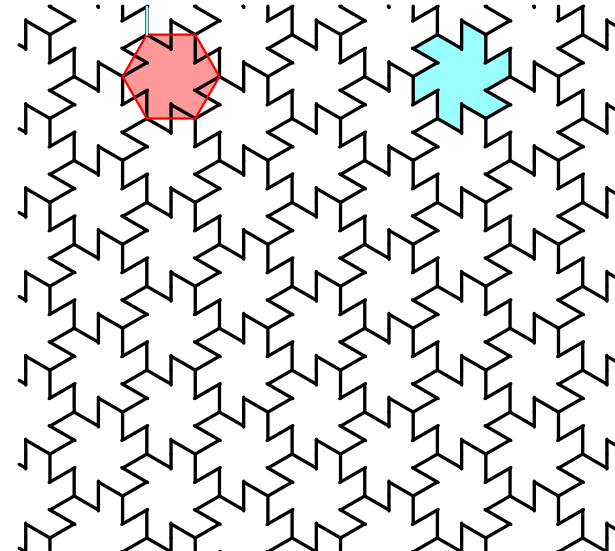
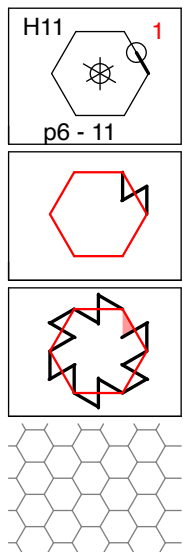
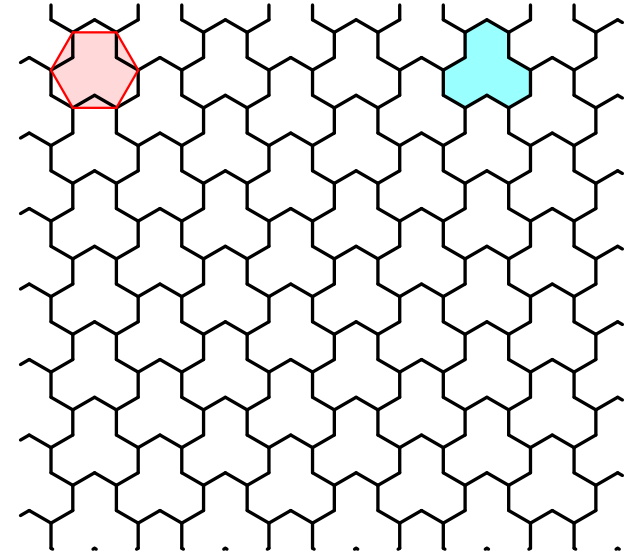
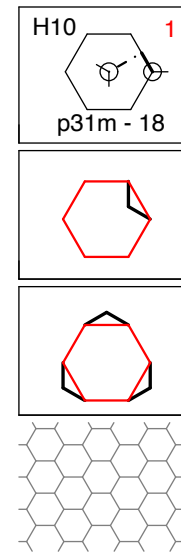
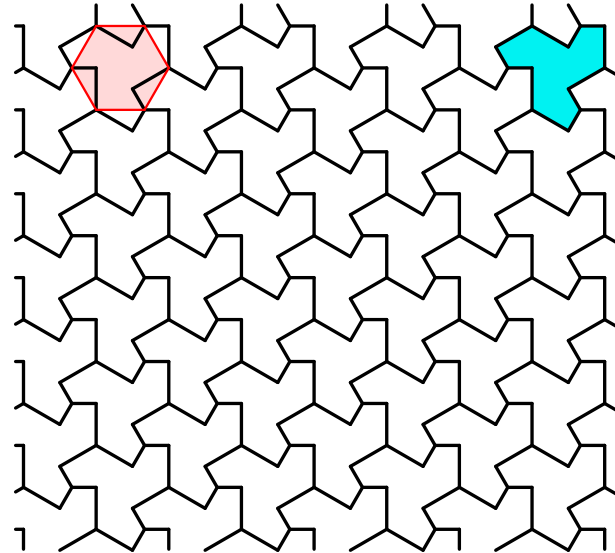
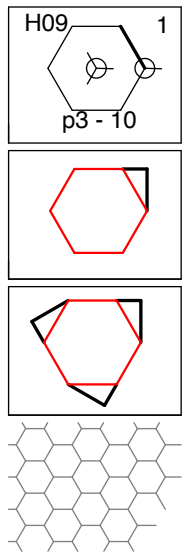
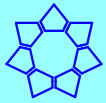


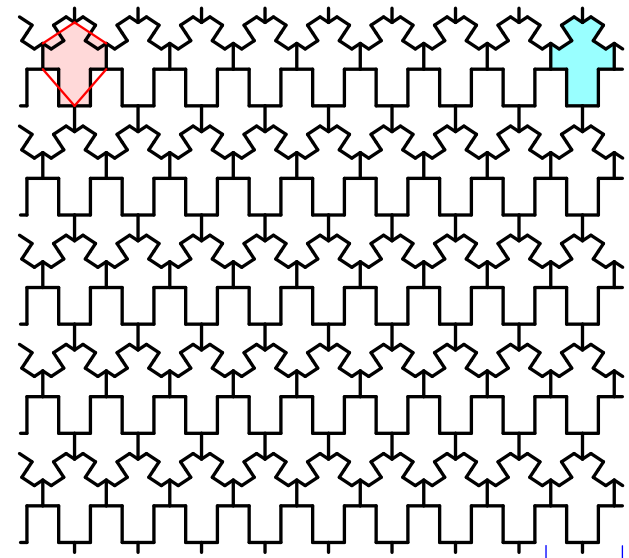
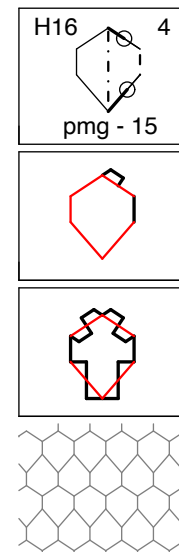
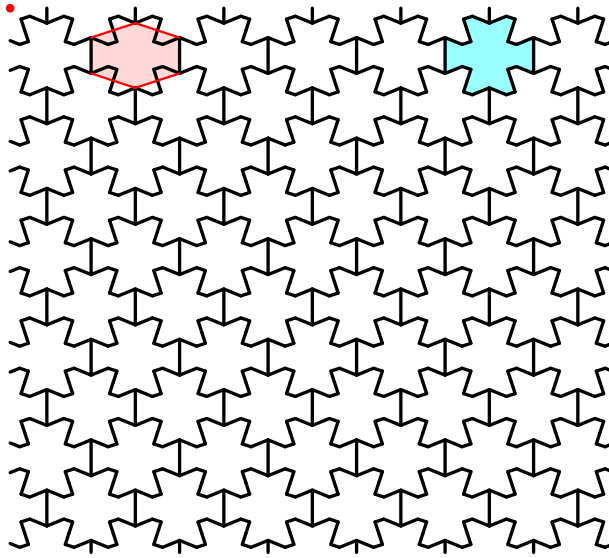
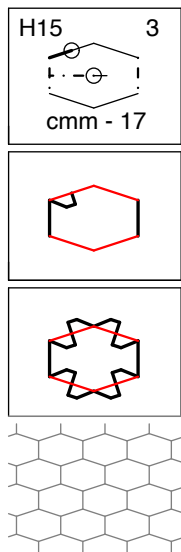
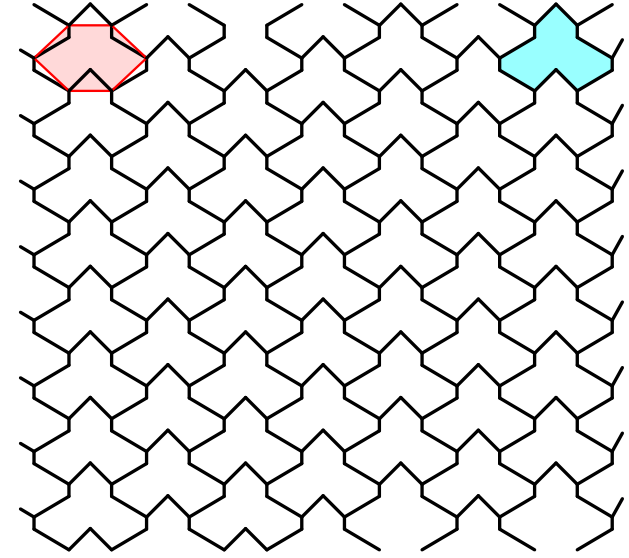
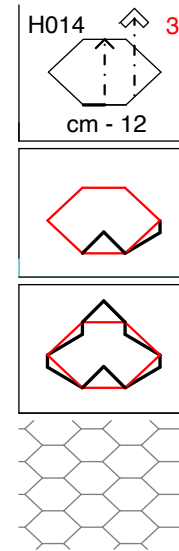
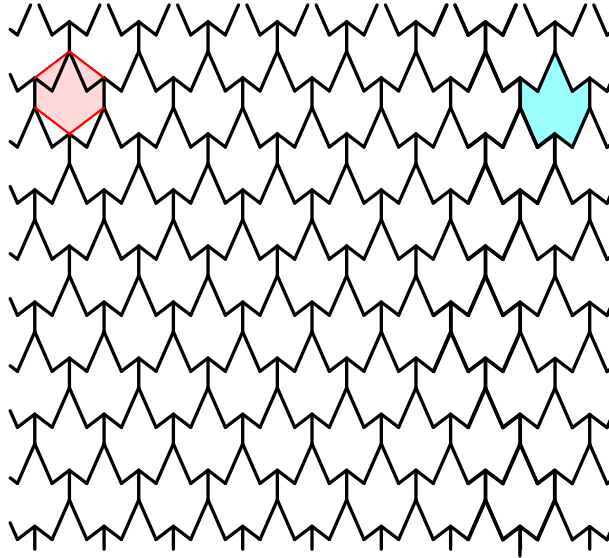
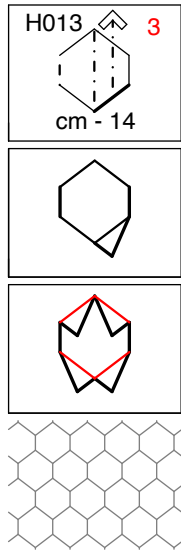
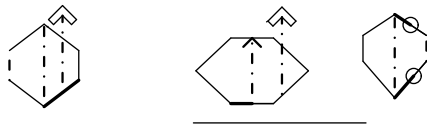
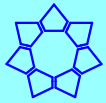


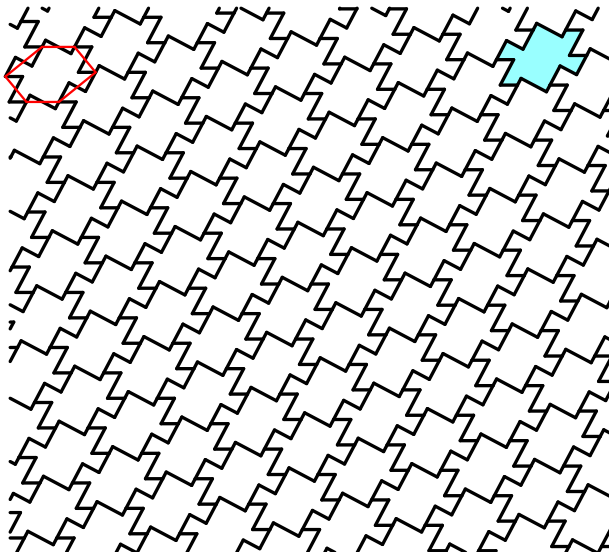
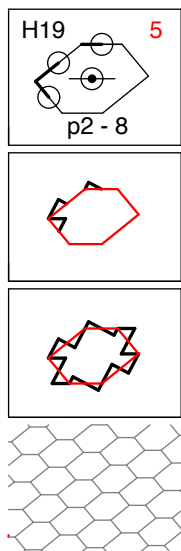
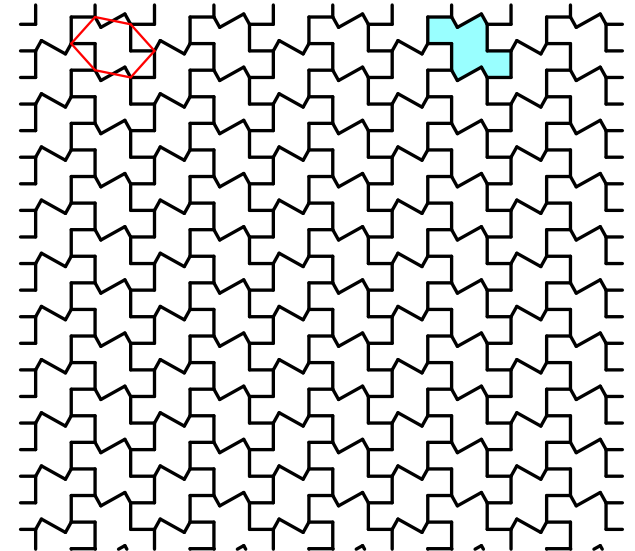
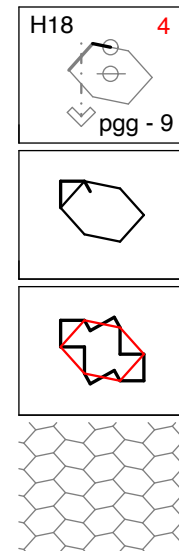
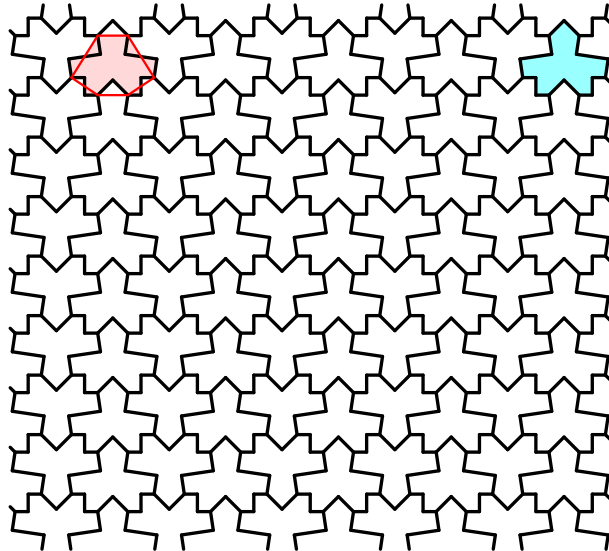
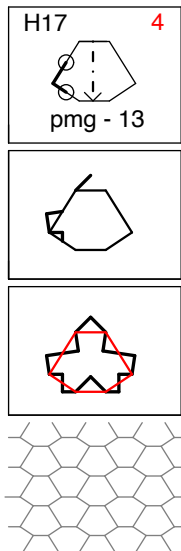
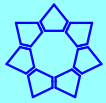


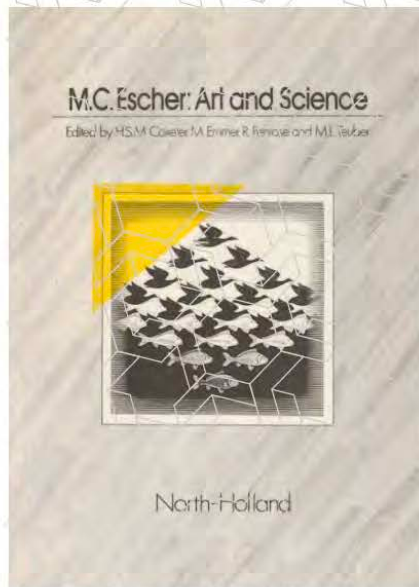








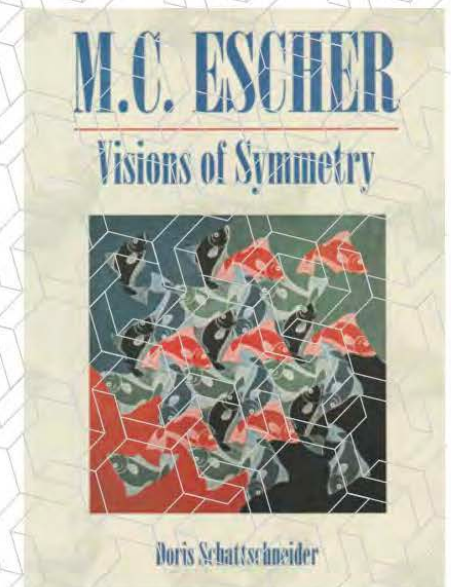




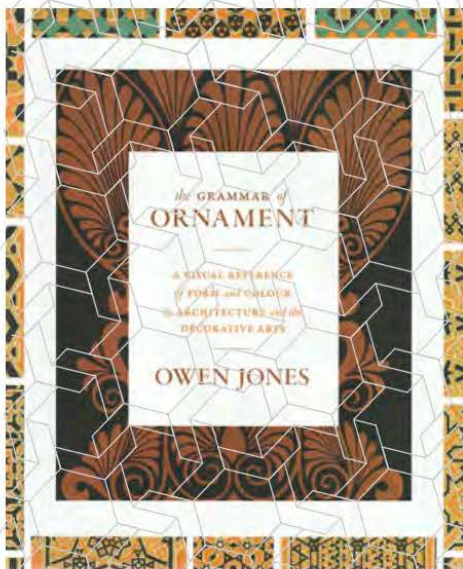
North-Holland, Amsterdam 1986



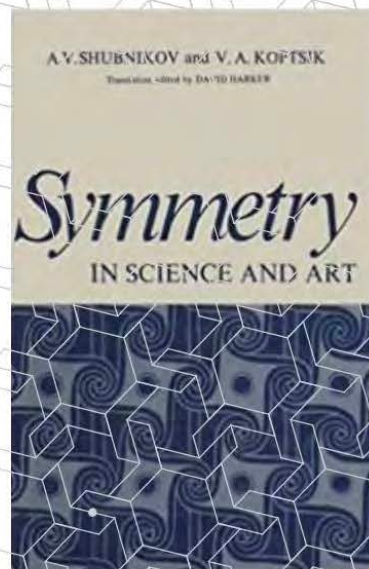
Harry N. Abrams Inc. New York 1989



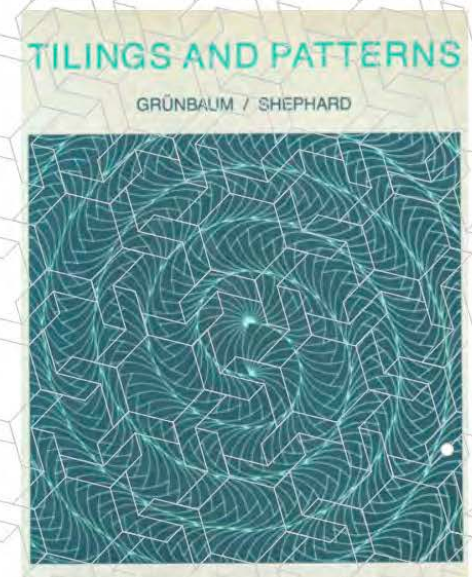
W. H. Freeman and Co. 1990



Princeton University Press 2016 (1856!)



Springer Verlag New York 2012 (1972)



W. H. Freeman and Co. 1987